CHRISTOFF RUDOLFF (1499 – 1543)

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In 1525, the first algebra book in German was published in Strasbourg. It was entitled *Behend und hübsch Rechnung durch die kunstreichen regeln Algebre, so gemeinicklich die Coß genannt werden* [Agile and pretty calculation through the ingenious rules of algebra, as the Coss are commonly called].

The word *Coss* comes from the Italian *Cosa* (*thing*, Latin: *res*) and stands for calculating with a variable.

The author of the work was CHRISTOFF RUDOLFF, whose exact dates of life are not known; nor is there a portrait of him. He himself gives Jauer as his place of birth, then in the Kingdom of Bohemia (now Jawor in Poland). Possibly he was born in 1499.

Between 1517 and 1521 RUDOLFF studied at the University of Vienna under HEINRICH SCHREIBER (Latinised as HENRICUS GRAMMATEUS).

SCHREIBER was the author of various arithmetic books, including *Libellus de compositione regularum pro vasorum mensuratione. Deque arte ista tota theoreticae et practicae* [A little book on the composition of the rules for the measurement of vessels. And about all this theoretical and practical art] from 1518. In this book SCHREIBER dealt with linear arithmetic, the rule of three (*Regel detri*), the rule of false position (*Regula falsi*), harmony (proportions in music), practical arithmetic for merchants, book-keeping and barrel measurement with a measuring rod.

SCHREIBER was the first to use the symbols "+" and "–" as arithmetic symbols for addition and subtraction. (The symbols "+" and "–" were invented by JOHANNES WIDMANN (1460-1498), who used them in a book about commercial arithmetic in 1489 to indicate surpluses and deficits.)

RUDOLFF probably remained in Vienna for the rest of his life and earned his living by selling his books and by teaching.

RUDOLFF's date of death can be deduced from information given by MICHAEL STIFEL. In his *Arithmetica integra* from 1544, STIFEL mentioned that CHRISTOFF RUDOLFF had died.

(Drawing © Andreas Strick)

RUDOLFF's book from 1525 comprises 206 double pages (*folia*). The first chapter deals with the written execution of the basic arithmetic operations and notes that *casting out nines* (or *sevens*) should always be carried out.

For multiplication, RUDOLFF mentions the multiplication tables as a basis, but points out that one actually only needs to master the multiplication tables for numbers less than or equal to 5.

Instead of the "big" factors 6, 7, 8, 9, one uses the "small" complementary numbers to 10, i.e. the numbers 4, 3, 2, 1 (in red). The units digit of the product is the product of the additions to 10, the tens digit is the difference between the tens digit of the first number and the addition to 10 of the second factor, if necessary taking into account the carry over from the ones digit.

This method also works the other way round (cf. the fourth example on the right – calculating with negative numbers).

9	1
8	2
9 – 2 = 7	1 · 2 = 2
7	2

7	3
7	3
7 – 3 = 4	3 · 3 = 9
4	9

6	4
7	3
6 – 3 = 3	4 · 3 = 12
3 + 1	2

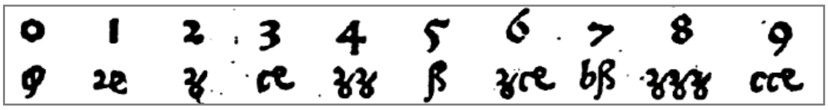
2	8
4	6
2 – 6 = -4	8 · 6 = 48
-4 + 4	8

RUDOLFF gives a remarkable hint concerning division by 10 or by 100: separate one or two digits from the right: 652 : 10 = 65|2 and 652 : 100 = 6|52 – possibly a first step towards decimal numbers.

In the second chapter, Rudolff deals with fractions: multiplying, reducing (canceling), extending, converting to a common denominator, adding and subtracting, comparing fractions. Interestingly, when dividing, the fractions are first converted to a common denominator, then the numerators are divided and the denominators are omitted.

The third chapter deals with *rule-of-three* problems and conversions of the monetary and measurement units of different countries; in the fourth chapter, one example each is used to explain how to extract a square root and a cube root in writing.

The actual "Coss" begins in the fifth chapter of the book with the introduction of variables. RUDOLFF uses separate symbols for the different powers:



The first symbol is a placeholder for the unit (*dragma* or *numerus*), today we would write x^0 for it, the second for the unknown quantity (*radix*, x), the third stands for the square of the unknown quantity (*zensus*, x^2), the fourth for the third power (*cubus*, x^3), etc.

For us it is unusual that RUDOLFF uses the dragma symbol throughout, e.g. he writes

for the equation $3x^2 + 4x = 20$ (“gleich” means “equal to”).

At the beginning of his explanations, RUDOLFF points out an important rule:

Only *denominations* (terms) of the same kind can be combined by addition or subtraction.

A calculation test can be done by inserting any numbers for the variables.

For the multiplication of the placeholders of different "orders", he first gives a combination table with all possible combinations, for example $x^2 \cdot x^3 = x^5$ (in our notation). In principle, he therefore describes the first exponent rule.

When calculating, sign rules (e.g. plus times minus is minus, etc.) as well as the distributive law are to be applied.

The sixth chapter deals with fractional expressions, for which the same rules apply as for ordinary number fractions. The following examples show his skill in dealing with the rules:

$$\frac{3}{4x^2}\cdot\frac{1}{4}x = \frac{3}{16x},\ \frac{3x+4}{5x^2-2x}\cdot\frac{4x-4}{5x^2+4} = \frac{12x^2+4x-16}{25x^4+20x^2-10x^3-8x},\ \frac{4x+5}{1x} : \frac{3}{3x-2} = \frac{12x^2+7x-10}{3x}.$$

In the seventh chapter, RUDOLFF deals with calculating with roots. He was the first in the history of mathematics to use the symbol √ for square roots.

Basically, he distinguishes between three types of numbers: *rational* or *skilful numbers*, which are square numbers such as the number 4, from which the root can be extracted, *communicants* or *mediocre numbers*, which are multiples of square numbers such as the number 8, from which – as we would say – the root can be partially extracted, and *irrational* or *awkward numbers*.

Representations of the form $a \cdot \sqrt{b}$ are not used by him; he writes this as $\sqrt{a^2 \cdot b}$. In the addition and subtraction of roots, one should examine the radicands for common factors. Thus, for example, in the case of $\sqrt{8}+\sqrt{18}$ decompose the radicands into $\sqrt{4 \cdot 2}+\sqrt{9 \cdot 2}$, then extract the root from the factors 4 and 9 respectively and add the results, so that the sum is 5, which, squared, is again drawn under the root. This gives: $\sqrt{8}+\sqrt{18}$ *makes* $\sqrt{50}$.

If this is not possible, square the sum or difference and then take the root again; thus, for example, with $\sqrt{7}-\sqrt{5}$ applying the binomial formula first gives $7+5-\sqrt{4 \cdot 7 \cdot 5}$, and $12-\sqrt{140}$ thus: $\sqrt{7}-\sqrt{5}$ *makes* $\sqrt{12-\sqrt{140}}$.

After corresponding remarks on third and fourth roots, in the tenth chapter RUDOLFF deals with the calculation of compound terms, such as $5+\sqrt{7}$ and $\sqrt{8}+\sqrt{6}$, which Rudolff calls a *binomium*, together with the respective *residuum* $5-\sqrt{7}$ or $\sqrt{8}-\sqrt{6}$. The interaction of *binomium* and *residuum* (and vice versa) in division is given special attention. From some of these *binomia* one got back to a *binomium* by taking the root, as for example from $14+\sqrt{180}$ to $3+\sqrt{5}$ or from $8+\sqrt{60}$ to $\sqrt{3}+\sqrt{5}$ – the others RUDOLFF calls *surdish* and *clumsy*.

The second book is divided into three sections. The first is about solving equations.

It begins by listing of the eight types of equations. For each of them he gives several examples (the exponents of the powers that occur increase by 1). Zero as a solution and negative solutions are ignored throughout. Instead of the equals sign in these equations, he writes the words "is equal" between the two terms. All these examples have 2 as a solution.

- *First Type:* $3x=6$; $4x^2=8x$; $5x^3=10x^2$; ...; $11x^9=22x^8$
- *Second Type:* $2x^2=8$; $3x^3=12x$; $4x^4=16x^2$; ...; $9x^9=36x^7$
- *Third Type:* $2x^3=16$; $3x^4=24x$; $4x^5=32x^2$; ...; $8x^9=64x^6$
- *Fourth Type:* $2x^4=32$; $3x^5=48x$; $4x^6=64x^2$; ...; $7x^9=112x^5$
- *Fifth Type:* $3x^2+4x=20$; $5x^3+6x^2=32x$; $7x^4+8x^3=44x^2$; ...; $6x^9+10x^8=44x^7$
- *Sixth Type:* $4x^2+8=12x$; $5x^3+9x=14\frac{1}{2}x^2$; $6x^4+10x^2=17x^3$; ...; $11x^9+15x^7=29\frac{1}{2}x^8$, also $2x^2+30=19x$; $3x^3+31x=21\frac{1}{2}x^2$; $4x^4+32x^2=24x^3$; ...; $9x^9+37x^7=36\frac{1}{2}x^8$
- *Seventh Type:* $4x+12=5x^2$; $5x^2+14x=6x^3$; $6x^3+16x^2=7x^4$; ...; $11x^8+26x^7=12x^9$
- *Eighth Type:* Equations that can be solved like types 5, 6 or 7; with x^4 instead of x^2 and x^2 instead of *x*, and correspondingly x^6 instead of x^3, x^8 instead of x^4, etc.

In the first edition of the book, RUDOLFF does not mention the fact that all equations of type 6 have one additional solution; he corrects this later.

The second section deals with the four operations (*cautelae*) that lead to the solution of an equation:

– *If terms of the same order occur several times in an equation, then combine them, if necessary by addition or subtraction on both sides of the equation.*

– *If terms occur in an equation that are preceded by a minus sign, then balance by addition.*

– *If there are root terms, then square them.*

– *If there are fraction terms on both sides, then cross-multiply.*

The third section of the second book contains 145 double pages with over 400 exercises with complete solutions (determining the variables, setting up the equation, transformations according to the above rules). Apart from the long description, the solutions hardly differ from those of the algebra exercises as we know them from today's textbooks.

Here is a small selection of the first type:

- If a number less than 10 is multiplied by 3, the product is 7 times as much greater than 10 as the number is less than 10.
- Of nine numbers in an arithmetical progression (growing over itself with equal transgression), the smallest is the number 4 and the sum is 48.
- Determine the length of a height in a triangle with sides 13, 14, 15.
- In a right triangle, one of the legs has length $3+\sqrt{18}$, the other leg and the hypotenuse together have length $9+\sqrt{162}$.

The 240 exercises on equations of the first type deals with problems that had already appeared in FIBONNACCI's collection of exercises and that could later be found in LEONHARD EULER's *Complete Guide to Algebra.*

The exercises are about gamblers who win or lose different amounts, about travellers who move towards each other from two towns. Then two men compare the contents of their purses, three men would like to buy a house or an entire village, acquire a horse or several horses, but need money from the others to do so. Further calculations follow on the salary of mercenaries to conquer a town, workers who are paid for days of work and money is deducted for days of absence, the income from a bridge toll has to be broken down again afterwards, grain is ground in three mills with different capacities at the same time and for the same length of time, companies are founded and profits divided, a musician visits three inns in succession and is paid for it, but has to pay for his meal each time, etc.

For systems of equations with two or even more than two variables, another variable is needed, which RUDOLFF calls *quantity* (and he writes this word out in the equation each time). The solution of these problems still seem somewhat complicated (from today's point of view), but only a few years later in STIFEL (see below) one can already find a solution variant with several variables.

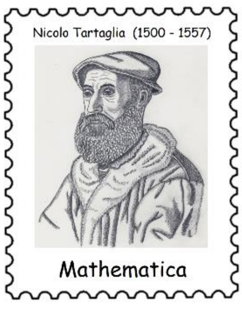
Among the exercises there is a problem with a negative solution, which RUDOLFF comments with *impossibility*.

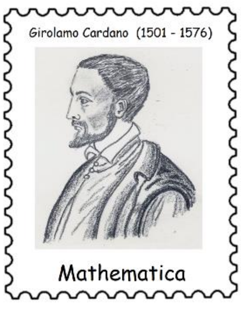
Overall, the high proportion of problems in which root terms occur is striking; this also applies to the other equation types, for example:

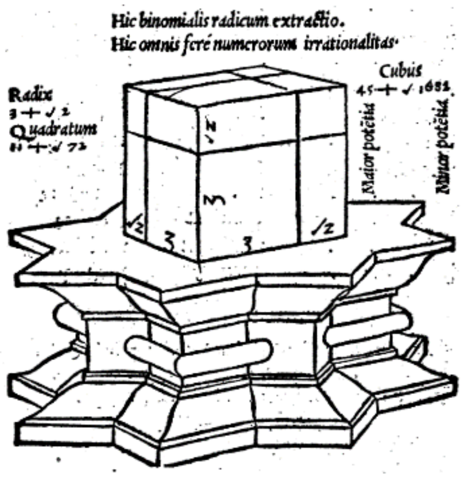
- If I multiply half of a number and one third of the number, then I get $36+\sqrt{1152}$.
- If I subtract 3 from the square of a number and also add 3 to the square of the number, then the product of the two results is $88-\sqrt{9408}$.
- One number is greater than another by $2+\sqrt{2}$. If I multiply them together the result is $36+\sqrt{1152}$.
- One person lends another 25 florins for two years with compound interest. After two years the latter pays back 49 florins.
- Two farmers sold oxen, one 30, the other several. For one ox, each of them received as many florins as the second farmer sold oxen. If you subtract the income of the first one from the income of the second one and take the third root from it, then you know how much money there was for one ox.
- You are looking for two numbers that add up to $10+\sqrt{18}$. If you multiply them, you get $25+\sqrt{338}$.
- One is asked how many weeks old he is. He answers: If you subtract 312 from a quarter of the number of weeks, then this is equal to the square root of the number of weeks minus 27.
- Two men have money, the second 4 florins less than the first. If you multiply the two fortunes together and square the product, the result is the same as if you multiply the third power of the larger amount by $5\frac{1}{3}$.

At the end of the book, RUDOLFF presents three problems that lead to a cubic equation, for the solution of which no systematic solution method was available at that time.

The cube shown at the end showed that he had an inkling of the approach discovered a few years later by TARTAGLIA, CARDANO and others.







Only one year after his algebra book, RUDOLFF published a second book with the title *Künstliche Rechnung mit der ziffer und mit den zal pfennigen* [Calculation with Indian/Arabic digits and on the calculating board] – a textbook for arithmetic students, merchants and craftsmen.

The first part (*Grundbüchlein*: Basic booklet) is comparable to that of his first work, in addition it also contains arithmetic with currency calculation.

In the second part (*Regelbüchlein*: Rule booklet), the *rule of three* and a method of practical arithmetic with units are explained and numerous examples are calculated. (*Exempelbüchlein*: Examples booklet).

The book had at least 17 editions by the end of the 16th century.

About ten years after RUDOLFF's death, *Die Coß Christoffs Rudolffs mit schönen Exempeln der* Coß *durch Michael Stifel gebessert und sehr vermehrt* [Christoff Rudolff's algebra with beautiful examples of the algebra improved and greatly extended by MICHAEL STIFEL] appeared in Königsberg.

STIFEL took over RUDOLFF's complete text and added numerous comments and supplements (e.g. on cubic equations), so that the volume was more than doubled and then amounted to 491 double pages.

In the preface, STIFEL praised RUDOLFF's work, which was no longer available, as: *being so clear and distinct that I could understand and learn it without any explanation* (*with the help of God*).

He also rejected the criticism that RUDOLFF had copied several examples from a manuscript that was in the Vienna library: *No one was harmed by taking over examples, rather this should be considered an honour, and besides, the purpose of a library is that everyone can use it.*

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https://www.spektrum.de/wissen/christoff-rudolff-verfasser-des-ersten-deutschen-algebrabuchs/2013418

Translated 2022 by John O’Connor, University of St Andrews

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