to the law of God. Every man must often choose between two courses, as to which is the more expedient; but this they hold to be a totally different thing. It is also urged against the probabilists, that they make the authority of *doctors*, or learned theologians, sufficient justification for a man's doing that which otherwise he would deem it unlawful to do; asserting that it will keep him safe at the judgment seat of God.

PROBABILITY, The Mathematical Theory of. Of all mathematical theories which can be made in any sense popular, this is perhaps the least generally understood. There are several reasons for this curious fact, of which we may mention one or two. *First.*—As by far the simplest and most direct elementary illustrations of its principles are furnished by games of chance, these have been almost invariably used by writers on the subject; and the result has been a popular delusion, to the effect that the theory tends directly to the encouragement of gambling. Nothing can be more false than such an idea. Independent of moral considerations, with which we have nothing to do here, no arguments against gambling can be furnished at all comparable in power with those deduced from the mathematical analysis of the chances of the game. *Second.*—In many problems, some of them amongst the easiest in the theory, the very highest resources of mathematics are taxed in order to furnish a solution. One reason is very simple. The solutions, however elementary, involving often nothing but the common rules of arithmetic, sometimes lead to results depending upon enormous numbers, and very refined analysis is requisite to deduce *easily* from these what would otherwise involve calculations, simple enough in character, but of appalling labour. Higher mathematics here perform, in fact, something analogous to *skilled labour* in ordinary manufactures. The simplest illustration of this is in the use of Logarithms (q. v.), which reduce multiplication, division, and extraction of roots to mere addition, subtraction, and division respectively. Powerful as logarithms are, analysis furnishes instruments almost infinitely more powerful. The large numbers which occur in probabilities are usually in the form of *products*, and we may exemplify the above remarks as follows.

To find the value of the product 1.3.5.7, no one would think of using anything but common arithmetic; but, if he were required to find the value of 1.3.5.7.9.........49, he would probably have recourse to logarithms, *merely to avoid useless labour of an elementary kind.* But in very simple questions in probabilities, it may be requisite to find (approximately) the value of a product such as 1.3.5.7.9.........23999—i. e., that of the first 12,000 odd numbers. No one in his senses would dream of attempting this by ordinary arithmetic, but it is the mere labour, not the inherent difficulty, which prevents him. Few would even attempt it by means of logarithms; for, even with their aid, the labour would be very great. It is here that the higher analysis steps in, and helps us *easily* to a sufficiently accurate approximation to the value of this enormous number. Thus, it appears that this objection to the study of the theory of probabilities is not applicable to their principles, which are very elementary, but to the mere mechanical details of the processes of solution of certain problems. *Third.*—There are other objections, such as the (so-called) religious one, that 'there is no such thing as chance,' and that 'to calculate chances is to deny the existence of an all-ruling Providence,' &c.; but, like many other similar assertions, these are founded on a total ignorance of the nature of the science; and, therefore, although pernicious, may be safely treated with merited contempt. The authors of such objections remind us of the Irishman who attempted to smash Lord Rosse's great telescope, because 'it is irreligious to pry into the mysteries of nature.'

It appears to us that the best method of explaining the principles of the subject within our necessarily narrow limits, will be to introduce definitions, &c., as they may be called for, in the course of a few elementary illustrations, instead of elaborately premising them.

First Case.—The simplest possible illustrations are supplied by the common process of 'tossing' a coin, with the result of 'head' or 'tail.' Put H for head, and T for tail. Now, the result of one toss, unless the coin should fall on its edge (which is practically impossible), *must* be either

H or T.

Also, if the coin be not so fashioned as to be more likely to fall on one side than the other (as, for instance, is the case with loaded dice), *these events are equally likely;* or, in technical language, *equally probable.* To determine numerically the likelihood or the probability of either, we must assign some numerical value to *absolute certainty.* This value is usually taken as *unity*, so that a probability, if short of absolute certainty, is always represented by a proper fraction. Suppose that p (a proper fraction) represents the probability of H, then evidently p is also the probability of T, because the two events are equally likely. But one or other *must* happen; hence, the sum of the separate probabilities must represent certainty. That is,

$$p + p = 1, \text{ or } p = \frac{1}{2}.$$

Thus we have assigned a numerical value to the probability of either H or T, by finding what proportion each bears to certainty, and assigning to the latter a simple numerical value.

Suppose, as a contrast, the coin to be an unfair one, such as those sometimes made for swindling purposes, with H on each side. Then we *must* have in one toss

H or H;

i. e., H is *certain*, or its probability is 1. There is no possibility of T, and therefore its probability is 0. Absolute impossibility is therefore represented by the numerical value of the probability becoming zero.

Second Case.—Suppose a 'fair' coin to be tossed twice in succession. The event *must* be one of the four—

H, H; H, T; T, H; or T, T.

Now all four are evidently equally likely; i. e., their probabilities are equal. But one of them *must* happen—hence the sum of their probabilities amounts to certainty, or 1. That is, each of the four cases has a probability measured by the fraction $\frac{1}{4}$.

Here we may introduce a new term. What are the *odds* against H, H? The answer is, the chance or probability of H, H is $\frac{1}{4}$; that is, *one* case in *four* is favourable, hence *three* are unfavourable, and the odds are said to be 3 to 1 against the event. *In general the odds against any event is the ratio of the probability that it will not, to the probability that it will, happen.*

Thus, in the first case above, the odds against H in one toss are even.

Third Case.—What is the chance of throwing both head and tail in two tosses of a coin? Remark that this is *not* the same question as, 'What is the chance of head followed by tail, in two tosses?' The latter question was answered in the *Second Case*, for the chance of H, T was there shewn to be $\frac{1}{4}$. The present event contemplates *either* H, T or T, H—and its probability is therefore $\frac{1}{4}+\frac{1}{4}$, or $\frac{1}{2}$, since each has the separate probability $\frac{1}{4}$. Or we may reason thus: Of the four possible cases of two tosses of a coin, *two* give both head and tail—*all four* are equally probable—hence the probability is 2 in 4, or 1 in 2; i. e., $\frac{1}{2}$.

Fourth Case.—What is the chance of throwing H in two tosses? Remark that this is *not* the same question as, 'What is the chance of H *once only* in two tosses?' The latter question is that of the Third Case merely put in a different form. Nor will it do to answer our question thus:

Chance of H in first throw $=\frac{1}{2}$.

Chance of H in second throw $=\frac{1}{2}$.

Therefore chance of H in two throws $=\frac{1}{2}+\frac{1}{2}=1$.

For by this reasoning it would appear that we *must* get head once at least in two throws; which is obviously absurd, for we may have T, T.

This very elementary example shews how delicate the reasoning in this subject is, and how liable one is to make (complacently) the most preposterous mistakes.

The error of the above process is introduced by the fact, that we have not considered that *if H be obtained in the first throw, our object is attained, and no second throw is required.* The correct work is this—

Chance of H in first throw $=\frac{1}{2}$.

If H come, the game is finished.

Chance of T in the first throw, in which case we must throw again, $=\frac{1}{2}$

Subsequent chance of H in second throw $=\frac{1}{2}$

Combining these, we have—

Chance of H at second throw *only* $=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$.

Add chance of H at first throw $=\frac{1}{2}$.

Sum, or chance of H in two throws $=\frac{3}{4}$.

A simpler method is this. The possible throws, all equally likely, are, as before—

H, H; H, T; T, H; and T, T.

The first three of these satisfy the requirements of the question; i. e., the required event has 3 chances in 4 in its favour, or its probability is $\frac{3}{4}$.

Fifth Case.—The chance of H in any one throw is $\frac{1}{2}$ (by *First Case*). The chance of H, H is $\frac{1}{4}$ (*Second Case*). Now $\frac{1}{4}=\frac{1}{2}\times\frac{1}{2}$; i. e., the chance of *the joint occurrence of two independent events*, at least in this simple case, *is the product of their separate probabilities.* Contrast this with the principle, already several times employed, that *the probability of an event which may arise from one of a number of causes* (*no two of which can coexist*), *is the sum of the separate probabilities.* Simple proofs of these statements, in all their generality, will now be given, along with various other important propositions.

(A.) If an event may occur in p ways, and fail in q ways—all being equally likely—the probability of its happening in one trial is $\frac{p}{p+q}$, and of its failing, $\frac{q}{p+q}$—and the odds in its favour are $p:q$.

The simplest way of conceiving this, and many other hypothetical cases, is to suppose *one* ball to be drawn from a bag which contains a number of balls, differing from each other in colour, or in some other quality not distinguishable by the touch. Suppose the bag to contain p white balls (W), and q black ones (B), and *one* ball to be drawn; what is the chance of its being white?

Here there are p chances in favour of a white ball being drawn, and q chances against it—these being *all equally likely*, or having *equal probabilities*—the chance of W is therefore p in $p+q$; i. e., is expressed by the fraction,

$$\frac{p}{p+q}.$$

The chance against W is q in $p+q$, or

$$\frac{q}{p+q}.$$

And the sum of these fractions is 1, or certainty, as it ought to be—for the ball drawn *must* be either W, or not W.

(B.) If an event may occur in p ways, and fail in q ways, all being equally likely—what are the chances of (a) its happening twice, (b) its happening the first, and failing the second, (c) its failing the first time, and happening the second, and (d) its failing twice, in two trials?

Taking the illustration in (A) above, we see that there are p independent ways of succeeding in the first case, and p in the second; hence, there are $p\times p$, or p^2 independent ways of *succeeding* twice. For any one of the first p may occur along with any one of the second. But the whole possible number of ways of experimenting twice is $(p+q)(p+q)$, or $(p+q)^2$; hence, the

Chance of (a) i.e. succeeding twice, is $\frac{p^2}{(p+q)^2}$.

Similarly, chance of (b) is $\frac{pq}{(p+q)^2}$.

" " " (c) is $\frac{qp}{(p+q)^2}$.

" " " (d) is $\frac{q^2}{(p+q)^2}$.

The sum of these is $\frac{p^2+2pq+q^2}{(p+q)^2}=1$, as it ought.

(C.) An attentive consideration of (B) shews us that when we have the independent probabilities of two events, the probability that they will jointly occur is the *product* of their separate probabilities.

Thus, for W, in first trial, chance is $\frac{p}{p+q}$.

" " second " " " $\frac{p}{p+q}$.

Whose product is $\frac{p^2}{(p+q^2)}$; the probability of W in each of two successive trials.

Again, for W, in the first trial, chance is $\frac{p}{p+q}$.

" B " second " $\frac{q}{p+q}$

Whose product is $\frac{pq}{(p+q)^2}$, which is found above to be the chance of WB. And so on.

(D.) This may be generalised as follows—the process will be evident to all who can understand the very elementary algebra employed:

$$\text{Certainty} = 1 = \frac{(p+q)^n}{(p+q)^n} = \frac{p^n + np^{n-1}q + \frac{n.\overline{n-1}}{1.2}p^{n-2}q^2 + \ldots\ldots + q^n}{(p+q)^n}$$

by the Binomial (q. v) Theorem of Newton. Now the *parts* of this expression—i. e.,

$$\frac{p^n}{(p+q)^n}, \frac{np^{n-1}q}{(p+q)^n}, \ldots\ldots\ldots\ldots, \frac{q^n}{(p+q)^n}$$

represent, obviously, the chances of W n times, W $n-1$ times and B once, W $\overline{n-2}$ times and B twice, ..., B n times, in n trials, where the *order* of occurrence is not considered.

If the *order* be considered, the chance of any arrangement, such as WBWWBBBW, for instance, is evidently

$$\frac{p\cdot q\cdot p\cdot p\cdot q\cdot q\cdot q\cdot p}{(p+q)^8} = \frac{p^4q^4}{(p+q)^8}.$$

But the chance of 4W and 4B in 8 trials, *without respect to order*, is as above, the term containing p^4q^4 in the expansion of $(p+q)^8$, divided by $(p+q)^8$—i.e.,

$$\frac{70\,p^4q^4}{(p+q)^8}.$$

To take a simple example: if there be 2W and 1B in a bag, and each ball be replaced immediately after drawing, the chance of W 4 times in succession is $\frac{2^4}{3^4} = \frac{16}{81}$.

Of the particular combination WBWB, the chance is $\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} = \frac{4}{81}$.

But the chance of W twice and B twice, *without respect to order*, is $6\,\frac{2^2.1^2}{3^4} = \frac{24}{81}$; the numerator of the fraction being the term of $(2+1)^4$ which contains the product $2^2.1^2$.

(E.) From the preceding results it is obvious that *the probability of the joint occurrence of any set of independent events is the product of their separate probabilities.*

(F.) We may vary the process by supposing that there are *several* bags, each containing some balls, which may be white or black; but the number in each bag, and the proportion of white to black, being any whatever. *One* ball only is to be drawn, what is the chance that it is W?

If n be the number of bags, the chance that the ball will be drawn from any particular bag is $\frac{1}{n}$ [see (A)]. And if in that bag there be p of W and q of B, the chance that W will be drawn from it is $\frac{p}{p+q}$ [see (A)].

Hence the chance that W is drawn, and from the particular bag, is,

$$\frac{1}{n}\cdot\frac{p}{p+q} \text{ by (E).}$$

And the whole chance that W is drawn is the sum of all the chances, $\frac{1}{n}\cdot\frac{p}{p+q}$, for each of the bags.

Thus, let there be 5 bags, containing, respectively, WB, WW, BB, WWB, WWW; our chance is found as follows: The chance of the ball being drawn from any particular bag is $\frac{1}{5}$, since all are equally likely to be chosen. Then, supposing the first chosen, the chance of W is $\frac{1}{2}$; if the third be chosen, the chance of W is 0, &c. Hence, on the whole, the chance of W is

$$\frac{1}{5}\cdot\frac{1}{2} + \frac{1}{5}\cdot 1 + \frac{1}{5}\cdot 0 + \frac{1}{5}\cdot\frac{2}{3} + \frac{1}{5}\cdot 1 = \frac{19}{30}.$$

(G.) Hence, *if an event may happen in consequence of any one of a set of causes, such that the action of one excludes that of the others; its probability is the sum of the products formed by multiplying the chance of the action of each cause by the chance that that cause, if operating, will produce the desired event.*

We might easily extend this very simple series of results, but our limits restrict us to an attempt to shew more the *extent* of the subject than the details of its application to any particular set of questions. We therefore reluctantly pass to the consideration of an *inverse* problem or two.

(H.) An event has occurred, which may have arisen from any one of a set of mutually exclusive causes: to determine the probability that any particular cause was the efficient one—the probability of the event's happening, when any particular one of the causes operates, being known.

As a simple example will shew us how to proceed in the most general case, take the 5 bags of (F) above. The chances of drawing W from them are, in order, $\frac{1}{2}$, 1, 0, $\frac{2}{3}$, 1. Suppose W *has* been drawn, what is the chance that it was drawn from any particular bag? It is obvious that the chance of W having been drawn from any particular bag is proportional to the chance that, if that bag had been selected, W would have been drawn from it. Hence, if p_1, p_2, p_3, p_4, p_5 be the chances that the several bags furnished the W actually drawn, we have

$$p_1 : p_2 : p_3 : p_4 : p_5 :: \frac{1}{2} : 1 : 0 : \frac{2}{3} : 1,$$

with the additional condition, that the ball *must* have been drawn from one of the bags, and therefore

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1.$$

From these, by elementary algebra, we have

$$p_1 = \frac{3}{19}, p_2 = \frac{6}{19}, p_3 = 0, p_4 = \frac{4}{19}, p_5 = \frac{6}{19}.$$

And a very simple application of algebra will easily conduct us to the general formula for any such case.

(I.) If the nature of a cause is known only by its results, we have an interesting case of simultaneous application of the direct and inverse methods.

Thus, a bag contains 3 balls, each of which may be either black or white. A ball has been drawn from it on two occasions—replacing before drawing—and on each of these occasions the ball was W. What is the chance that a third drawing will give a *black* ball?

The contents of the bag are obviously one of the following—viz., W,W,W; W,W,B; or W,B,B—since it contains *one* W at least. Now, if WWW be the contents, the probability of the observed event (two W in succession) is $1 \times 1 = 1$.

If W,W,B, $\quad \frac{2}{3}\times\frac{2}{3} = \frac{4}{9}$.

If W,B,B, $\quad \frac{1}{3}\times\frac{1}{3} = \frac{1}{9}$.

Hence the probabilities that these are, respectively, the contents of the bag are as $1 : \frac{4}{9} : \frac{1}{9}$, or as $9:4:1$; and are therefore $\frac{9}{14}$, $\frac{4}{14}$, and $\frac{1}{14}$ respectively, since their sum must be 1 or certainty.

Now for the chance of B in the third drawing; if WWW be the contents $\left(\text{of which the chance is } \frac{9}{14}\right)$, the chance of B is 0. Hence we have one part of the chance for B, viz. $\frac{9}{14} \times 0 = 0$. Similarly, the other parts are $\frac{4}{14} \times \frac{1}{3} = \frac{4}{42}$, and $\frac{1}{14} \times \frac{2}{3} = \frac{2}{42}$. The whole chance of B in the third drawing is therefore

$$0 + \frac{4}{42} + \frac{2}{42} = \frac{1}{7}.$$

As exercises on the above principles, we will take first a few simple questions from Life Assurance, the subject to which, above all others, the elementary theory of Probability has been of the most indispensable service. We purposely choose the very simplest that the subject can furnish, but they are quite sufficient to shew the great value of the theory.

A Table of Mortality (q. v.) gives the numbers alive at each successive year of their age, out of a given number of children born. If A_n and A_{n+1} be the numbers in the table corresponding to the n^{th} and $\overline{n+1}^{\text{th}}$ years of age; the inference from the table is, that, of A_n individuals now alive, and of n years of age, A_{n+1} will live one additional year at least. Hence, the chance that any one of them die during the year is

$$\frac{A_n - A_{n+1}}{A_n}.$$

Call this $1 - p$, then p is the chance that any one of them will survive the year.

Questions. Of two individuals, one n years old, and the other n_1, what are the chances that

(*a.*) Only one lives a year?
(*b.*) One, at least, lives a year?
(*c.*) Both do not live a year?

Calling the individuals A and B, the chance of A living out the year is p, and the chance of his dying within the year is $1 - p$. For B these are p_1 and $1 - p_1$. Hence

(*a.*) A lives and B dies—chance $p(1 - p_1)$.
B lives and A dies—chance $(1 - p)p_1$.

Hence answer to (*a*) is . . $p + p_1 - 2pp_1$.

(*b.*) This includes, in addition to the conditions of (*a*), the chance that *both* survive, which is pp_1.

Hence answer to (*b*) is $p + p_1 - pp_1$.

(*c.*) In this case the chance that both *do* live a year is pp_1. Hence chance of (*c*) is $1 - pp_1$.

As another very instructive example, let us take the question,

'In how many throws of a die is it even betting that an ace will be thrown?'

This may, of course, be worked directly, proceeding in the following manner:

$$\text{Chance of ace in first throw} = \frac{1}{6}.$$

Then, remembering that there is no second throw unless the first fails,

Chance of ace in second throw $= \frac{5}{6} \cdot \frac{1}{6}$; and so on.

Hence the odds against ace in 1 throw are 5 : 1.
" " " 2 throws 25 : 11;

and so on. But great care is requisite in this mode of working the problem.

The simplest procedure is this:

Chance *against* ace in	1 throw	$\frac{5}{6}$.
" " "	2 throws	$\frac{25}{36}$.
" " "	3 "	$\frac{125}{216}$.
" " "	4 "	$\frac{625}{1296}$.

Hence odds against ace in	1 throw	5 : 1.
" " "	2 throws	25 : 11.
" " "	3 "	125 : 91.
" " "	4 "	625 : 671.

That is, the odds are considerably *against* ace occurring in three throws, being about 11 to 8; while in four they are slightly in its favour, as 29 : 27 nearly. One is sure, therefore, of winning in the long run, if he can get any one to give him repeatedly an even bet against ace appearing in four throws of a die.

It is to be observed that when we say '*in the long run*,' we mean that the most likely event may not be that which will happen in the first trial, nor perhaps for many trials (because, unless its probability is 1 or certainty, it is, of course, *possible* that it may never occur). But what is certain is this, that if a sufficient number of trials be made, we can have any amount of probability (short of certainty) that the ratio of the number of successful trials to the number of failures, will be in the ratio expressed by the odds in favour of success in any one trial.

And this introduces us to another department of the theory of Probabilities, what is called *Expectation.* We begin with a simple case, not involving what is called *Moral Expectation*, to which the next example will be devoted.

Suppose A, B, and C have made a pool, each subscribing £1; and that a game of *pure chance* (i. e., not dependent on skill) is to be played by them for the £3. What is (previous to play) the value of the expectation of each? By the conditions, all are equally likely to win the pool, hence its contingent value must be the same to each; and, obviously, the sum of these values must represent the whole amount in question. The worth of the expectation of each is therefore £1. That is, if A wishes to retire from the game before it is played out, the fair price which B or C ought to pay him for his share is simply £1. But this is obviously $\frac{1}{3} \times$ £3; i. e., the value of the pool multiplied by his chance of getting it. Here we have taken an extremely simple case, because we have not room for the general proof (though it is closely analogous to that just given) that

The value of a contingent gain is the product of the sum to be gained into the chance of winning it.

So far, it has been assumed that the payment of his stake (which may be wholly lost) has not *morally* affected the position of any of the players; i. e., that the stake is a sum whose loss would in nowise embarrass him. And it is only with such cases that the strict mathematical theory can deal; for we cannot estimate with mathematical accuracy the value of the stake as depending on the fortune (the *possessions*, not the *luck*) of the player. The attempts which have been made to supply this apparent deficiency in the theory have, of course, not been very generally accepted. Still there is no doubt that two men of very unequal fortunes are placed in very different circumstances when they have subscribed equal sums to a pool which they have equal chances of gaining. The most commonly received method of *approximating* to a solution of such a question (for it is obvious that here we have left mathematical certainty behind) is that proposed by Daniel Bernoulli; which is, that the value of a small gain, or the inconvenience of a small loss, is directly proportional to the amount of the gain or loss (which is probably correct), and inversely proportional to the fortune of the person affected (which may be nearly true, except in very extreme cases). The application of this hypothetical principle necessitates, in general, the use of the integral calculus; but, to shew the *mathematical folly* of gambling, we quote one of Bernoulli's results.

A, whose whole fortune is £100, bets £50 even with B on an event of which the chance is $\frac{1}{2}$. What is the moral value of A's fortune after making the bet (and before it is decided)? By applying the above method, he finds it to be £87. Thus A, by making the bet, has depreciated by 13 per cent. the value of his property. This is an extreme case, of course; and the method employed in obtaining the result is questionable; still, it is certain that no legitimate method could shew that A had otherwise than impaired his fortune by entering upon any such transaction. This, of course, is on the supposition that the bet is a *fair* one; if A be a swindler, and get from B more than the proper odds against the event, he may, of course, improve to any extent the value of his fortune. But such would be a question of flats and sharpers, not a question of probability.

A very excellent example of *moral* as distinguished from *mathematical* probability is furnished by the famous 'St Petersburg Problem.'

A and B play at heads and tails. A is to pay B £2 if H comes at the first throw, £4 if at the second and not before, £8 if at the third and not before; and so on, doubling each time. What should B pay (before the game) for his expectation?

Applying the mathematical method, we see that

$$\text{Chance of H at first throw} = \frac{1}{2};$$

in which case B gets £2, of which the contingent value is $\frac{1}{2} \times £2 = £1$.

$$\text{Chance of H at second throw, and } \textit{not} \text{ before} = \frac{1}{4};$$

when B is to get £4, whose value is therefore $\frac{1}{4} \times £4 = £1$.

$$\text{Chance of H at third throw, and not before} = \frac{1}{8};$$

contingent value of B's £8 is therefore $\frac{1}{8} \times £8 = £1$.

And so on, for ever.

Hence B's expectation (mathematical) is £1 + £1 + £1 + &c. for ever, or an infinite sum. Now it is obvious that no man, in his senses, would pay even a moderately large sum for such a chance. Here the *moral* expectation comes into play; but the mathematical solution is perfectly correct, if we interpret it properly. *It does not attempt to tell what will be the actual result in any one game—this is pure chance*—but it tells us what will be the average to which the results of larger and larger numbers of games *must* continually tend. In other words, if B had an inexhaustible purse, he might safely pay *any* amount to A before each game, *and be sure of winning in the long run*, after an indefinitely great number of games were played. But this, though theoretically exact, is not applicable to mundane gambling—where limited purses and limited time circumscribe the field requisite for the proper development of the mathematical result.

Before quitting this part of the subject, we may give a couple of instances in which the mathematical theory may be easily tested by any one who has a little leisure. One of these we will develop at length, as a final instance of the simple calculations generally involved.

'To find the chance of throwing any given possible number with two dice.'

As the faces of the dice are numbered from 1 to 6—the smallest throw is 2, and the greatest 12.

In one throw, the chances are—

$$\text{For } 2 = 1 + 1;\ \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36};$$

the probabilities being *multiplied* (E) because the events are independent. For

$3 = 1 + 2$, or $2 + 1$; $\frac{2}{36}$:

$4 = 1 + 3$, $2 + 2$, or $3 + 1$; $\frac{3}{36}$:

$5 = 1 + 4$, $2 + 3$, $3 + 2$, or $4 + 1$; $\frac{4}{36}$:

$6 = 1 + 5$, $2 + 4$, $3 + 3$, $4 + 2$, or $5 + 1$; $\frac{5}{36}$:

$7 = 1 + 6$, $2 + 5$, $3 + 4$, $4 + 3$, $5 + 2$, or $6 + 1$; $\frac{6}{36}$:

Then, in the inverse order—

$8 = 2 + 6$, $3 + 5$, $4 + 4$, $5 + 3$, or $6 + 2$; $\frac{5}{36}$:

and so on—the fact being that if we read the *lower* sides of the dice when the throw is 8, they will give 6, and so on—the sum being always 14.

The mathematical expectation for any one throw is therefore

$$\frac{1}{36} \cdot 2 + \frac{2}{36} \cdot 3 + \frac{3}{36} \cdot 4 + \frac{4}{36} \cdot 5 + \frac{5}{36} \cdot 6 + \frac{6}{36} \cdot 7$$
$$+ \frac{5}{36} \cdot 8 + \frac{4}{36} \cdot 9 + \frac{3}{36} \cdot 10 + \frac{2}{36} \cdot 11 + \frac{1}{36} \cdot 12$$

In all—

$$\frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12),$$

$$\text{or } \frac{1}{36} \cdot 252 = 7.$$

The meaning of this is, *not* that we shall probably throw seven the first time, nor second, nor perhaps for many throws; but that if we throw a number of times, add the results, and divide by the number of throws, the final result will be more and more nearly equal to seven, the greater be the whole number of throws. It is very instructive to make the experiment, say on 100 throws of two dice, as in backgammon. If the mathematical result be not closely verified by such a trial, *the dice are loaded*; or, at least, are ill-made.

Another illustration, and a very excellent one, is furnished by the following theorem.

If the floor be ruled with equidistant parallel lines, and a straight rod, whose length is equal to the distance between any two contiguous lines, be dropped upon it at random, the chance of its falling on one of the lines is $\frac{2}{\pi}$, where π is the ratio of the circumference of a circle to its diameter (see QUADRATURE OF THE CIRCLE). The deduction of this result from the theory of Probabilities requires the use of the integral calculus, and cannot be given here; but we may put the above theorem to the test of practice in the following way. Let the rod be tossed a number of times, then the greater this number, the more nearly shall we have

$$\frac{\text{Twice number of throws}}{\text{Number of times the rod falls on a line}} = \pi$$
$= 3 \cdot 14159$, &c.;

and therefore, by simply continuing this process long enough, we may obtain as accurate a value as we choose of the ratio of the circumference to the diameter of a circle.

To shew how the theory of Probabilities would tend, if generally known, to the discouragement of gambling, would require a treatise—as every species of game would have to be treated—we shall therefore only take one case, about as bad a one as can be. This is when a man makes a 'book' on a horse-race, so as to 'stand to win,' whatever be the result of the race. This is, of course, immoral; for, as it can make no matter who accepts his bets, suppose them all taken by one individual. The latter must therefore have been 'done' into a complex transaction by which he is *certain to lose.* The method of making such a 'book' is simple enough; it consists mainly in betting *against* each horse. Thus, if three horses, A, B, C, are to start, and he can get the following bets taken—

£4 : £3 against A,
£5 : £4 " B,
£6 : £5 " C;

his book stands thus:

If A win, he wins £4 + £5 − £4 or £5.
" B " " £5 + £3 − £5 " £3.
" C " " £4 + £3 − £6 " £1.

Now, to examine this case, suppose the *correct* odds to have been laid against A and B, what ought in fairness to be the odds as regards C?

Chance of A winning is . . $\frac{3}{7}$.

" B . . . $\frac{4}{9}$.

Chance of A or B winning $= \frac{3}{7} + \frac{4}{9} = \frac{55}{63}$.

Hence, chance of C winning $= \frac{8}{63}$; and therefore the legitimate odds against C are 55 to 8, whereas our betting-man has got a *fool* to accept 6 to 5.

The true cause of the detestation which attaches to gambling, is not so much the ruin, insanity, suicide, &c., in which it not unfrequently ends, as the fact, that a gambler's work in no case increases the wealth or comfort of the state; all it can effect is a more or less rapid and dishonest transfer of these from one state of distribution to another. It is as useless, so far as regards production, as the prison-crank.

There is a common prejudice as to '*runs of luck*,' which are popularly supposed *not* to be compatible with the mathematical theory. This, also, is a complete delusion. To take a very simple case, the reader will easily see that, if he writes down all the possible cases which may occur in six tosses of a coin, the odds are 19 : 13 in favour of a run of three at least.

To give an instance of the principle of interpretation which we have several times above applied to the mathematical result—viz., that the greater the number of trials, the more nearly will the average result of these trials coincide with it—let us recur to heads and tails. Suppose a coin tossed ten times, and let H^n stand for H *n* times, then we have

$$1 = \left(\frac{1}{2} + \frac{1}{2}\right)^{10}$$
$$= \frac{1}{2^{10}}\left(1 + 10 + \frac{10 . 9}{1 . 2} + \frac{10 . 9 . 8}{1 . 2 . 3} +, \text{\&c.}\right)$$

of which the terms are [as in (D)] the probabilities of H^{10}, H^9T, H^8T^2, &c. respectively; the order not being taken account of. The largest term is the 6th, and its value is $\frac{252}{2^{10}} = \frac{252}{1024}$, or about $\frac{1}{4}$. This is the chance of H^5T^5, without regard to order, in ten throws. Although the most probable result, inasmuch as the chances of H^6T^4 and H^4T^6 are each about $\frac{1}{5}$ only, and those of the other possible combinations much smaller—yet it has not a very large chance. But the chance of a result *not deviating much from the most probable one*, is very much larger: in the above case, the chance of having not less than 3H, and not less than 3T, is as much as $\frac{912}{1024}$. But this tendency of the bulk of the results to coincide very closely with the most probable one, is much more evident as we take a greater and greater number of trials. Thus, in 100 trials with the coin, we have—

$$1 = \left(\frac{1}{2} + \frac{1}{2}\right)^{100}$$
$$= \frac{1}{2^{100}}\left(1 + 100 + \ldots + \frac{100 . 99 . 98 \ldots\ldots 51}{1 . 2 . 3 \ldots\ldots 50} +, \text{\&c.}\right)$$

[*Now* we begin to see how the higher analysis comes in. Who is to work out by common arithmetic the value of the fraction $\frac{100 . 99 . 98 \ldots\ldots\ldots 51}{1 . 2 . 3 \ldots\ldots\ldots 50}$? Some calculating boy *might*, with no very enormous labour—but, wait a moment, we may wish to have the result of a million of trials, and what calculator (arithmetical) will tell us the value of

$$\frac{1{,}000{,}000 \times 999{,}999 \times \ldots\ldots\ldots \times 500{,}001}{1 \times 2 \times \ldots\ldots\ldots \times 500{,}000}?]$$

In this case, the most probable result is $H^{50}T^{50}$, without regard to order, but its chance is only about $\frac{2}{25}$. [The exact value is $\frac{1}{2^{100}} \cdot \frac{100 . 99 . 98 \ldots\ldots\ldots 51}{1 . 2 . 3 \ldots\ldots\ldots 50}$] Had there been 1000 throws, the chance of $H^{500}T^{500}$ (the most likely combination) would have been about $\frac{1}{38}$. But, as the number of throws increases, the number of terms grouped close to the largest in the expansion, and whose sum far exceeds that of all the rest, becomes a smaller and smaller fraction of the entire number of terms. Hence the chance that in 1000 tosses there should not be more than 600 nor less than 400 H, is much greater than that of not more than 60 nor less than 40 H in 100 throws; and so on.

Thus it is that all our statistical results, say the ratios of the numbers of births, marriages, suicides, &c. to the whole population—or that of the male to the female births—or that of the dead letters to the whole number posted, &c.—though perhaps never the same in any two years, yet fluctuate between very narrow limits. And thus it is that the theory of Probabilities has been the means of solidly establishing, beyond almost the possibility of failure, when properly applied, the inestimable securities afforded by life-assurance.

Another very important application of the theory is to the deduction of *the most probable* value from a number of observations (astronomical, meteorological,

&c.), each of which is liable to error. We may confidently assert that, but for this, astronomy could not have taken the gigantic strides by which it has advanced during the present century. But the '*Method of Least Squares*,' as it is called, which is furnished for this purpose by the theory of probabilities, is far beyond the scope of elementary mathematics, and can therefore only be referred to here. Its fundamental features may be seen in the above process of determining the probability that the result of a number of trials shall lie within certain limits on each side of the most probable result.

The theory of Probabilities has been applied to many other important questions, of which we may mention only two—the value of evidence, and the probability of the correctness of the verdict given by various majorities in a jury. But for these, and for the further development of what we have given above from the simplest points of view, we must refer to the various treatises on the subject. Of these, the most accessible to an English reader are the very valuable works of Galloway and De Morgan. Poisson, Gauss, and especially Laplace, have also treated the subject in the most profound manner. But the difficulty of understanding Laplace's great work is such, that few have ever mastered it completely; and it is therefore particularly satisfactory that the late Professor Boole, in his *Laws of Thought*, has shewn how to dispense with a great part of the analysis which renders Laplace's work so formidable.

PRO'BATE COURT is a court created in England in 1858, in lieu of the old Prerogative Courts, to exercise the exclusive jurisdiction in all matters touching the succession to personal estate. The rules on which its jurisdiction is founded are, that whenever a man dies he must either leave a will or not. If he leave a will, then it must be produced and verified, so as to demonstrate to all parties interested that it is an authentic will, and has been duly executed and signed in presence of witnesses, and therefore that the right to the personal estate is vested in the executors named by the will. The will is sworn to by the witnesses, on being produced; and if the evidence is satisfactory, it is registered, and the original deposited in the court, when copies are made. This process is called proving the will, and the act of court is called the probate of the will. If there is no will, then the rule being, that the personal estate devolves on the next of kin and widow, if any, it is necessary that an application be made to the court to appoint one of the next of kin to be the administrator, and take charge of the payment of debts. This is called taking out administration, and the act of the court appointing the administrators is called letters of administration. Numerous difficulties often arise as to irregularities in the making of wills and as to the party entitled to administration, and it is the function of the Court of Probate to dispose of these.

PROBOSCI'DEA, a section of *Ungulata*, of which the characters are given under *Pachydermo*, contains one recent and two fossil genera, *Elephas* (see ELEPHANT), *Mastodon* (q. v.) and *Dinotherium;* so that the P. seem not to have been numerous at any period of the world's history. Notwithstanding the great size of these creatures, comparative anatomists have pointed out various resemblances in their dentition, osteology, &c., to rodents.

PROBO'SCIS MONKEY. See NASALIS.

PRO'BUS, MARCUS AURELIUS, Emperor of Rome, was born at Sirmium, in Pannonia. His father, Maximus, served first as a centurion, and afterwards as a tribune in the Roman army, and died in Egypt, leaving to his only son a good name and a moderate income. P. early entered the army, and had the good fortune to attract the favourable notice of the Emperor Valerian, who elevated him before the legal period to the rank of tribune. His subsequent conduct justified his rapid promotion, for he greatly distinguished himself against the Sarmatians on the Danube, and subsequently in Africa, Egypt, Asia, Germany, and Gaul, winning golden opinions from Valerian's successors, Gallienus, Claudius II., Aurelian, and Tacitus. By the last-named emperor, he was appointed governor of the whole Asiatic possessions of Rome, and declared to be the chief mainstay of the Roman power; and such was the zealous attachment evinced for him by his soldiers, whose respect and love he had equally won by his firm discipline, by his care in providing for their wants and comforts, and his liberality in the distribution of plunder, that, on the death of Tacitus, they forced him to assume the purple; and his rival, Florianus, having been removed, P. was enthusiastically hailed emperor by all classes (276 A.D.). His brief reign was signalised by brilliant and important successes; the Germans, who, since Aurelian's time, had made Gaul almost a part of Germany, were driven out with enormous slaughter, pursued into the heart of their own country, compelled to restore their plunder, and to furnish a contingent to the Roman armies. Pursuing his victorious career, P. swept the inimical barbarians from the Rhætian, Pannonian, and Thracian frontiers, and forced Persia to agree to a humiliating peace. Various aspirants to the imperial purple were also put down. On his return to Rome, P. celebrated these fortunate achievements by a triumph, and then, the external security of the empire being established, devoted himself to the development of its internal resources. The senate was confirmed in its privileges; liberal encouragement was given to agriculture; numerous colonies of barbarians were established in thinly-peopled spots, that they might adopt a civilised mode of life; and all branches of industry were protected and promoted. But P. was at a loss what to do with his army, as the Romans had now no enemies either at home or abroad; and fearing that their discipline would be deteriorated by a life of inactivity, he employed the soldiers as labourers in executing various extensive and important works of public utility. Such occupations, considered as degrading by the soldiers, excited among them the utmost irritation and discontent; and a large body of troops, who were engaged in draining the swamps about Sirmium, giving way to these feelings, under the excitement produced by the presence of the emperor, murdered him, 282 A.D. P. possessed great military genius, combined with equal administrative talent, and added to these a wisdom, justice, and amiability equal to that of Trajan or the Antonines.

PROCAME'LES, an extinct genus from the Miocene of North America, discovered by Hayden in Nebraska, near the recent camels. Three species are known; one as large as the camel.

PROCE'SSION OF THE HOLY GHOST, that doctrine regarding the Third Person of the Blessed Trinity which teaches that as the Son proceeds (or is born) from the Father, so the Holy Ghost proceeds (or emanates) from the Father and from the Son, but as from one principle. The question of the origin of the Holy Ghost was not distinctly raised in the early controversies, which fell chiefly upon the Second Person. In the Creed of Nicæa, no allusion whatever is made to the subject; and in the Creed of Constantinople, the Holy Ghost is said simply to 'proceed from the Father.' Nevertheless, this was