ABU BAKR AL-KARAJI (953 – 1029) & **IBN AL-SAMAW'AL** (1130 – 1180)

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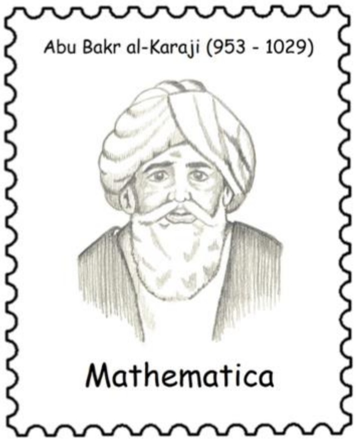

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All we know about the mathematician ABU BAKR IBN MUHAMMAD IBN AL-HUSAYN AL-KARAJI is that he spent many years of his life in Baghdad. Whether he came from Baghdad itself (one of the suburbs is called Karkh) or from the city of Karaj in Iran can no longer be clarified.

Only a few of AL-KARAJI's works have survived, among them an arithmetical manuscript with everyday arithmetical problems and a manual for civil servants, as well as the manuscript al-*Fakhri fi l-gabr wa-l-muqabala* (Wonderful things about arithmetic), consisting of two parts, which he dedicated to his ruler, the vizier AL-FAKHRI.

While the first part mainly contains tasks that can also be found in DIOPHANTUS and ABU KAMIL (850 – 930), the second part of the work is considered by posterity to be the first truly algebraic writing.

ABU KAMIL had already taken decisive steps towards abstract algebra in a commentary on AL- KHWARIZMI's *Al Kitab al-muhtasar fi hisab al-gabr w-al-muqabala* by establishing rules for products of sum terms and explaining how root terms can be transformed, e.g.

$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ and $\sqrt{a / b} = \sqrt{a} / \sqrt{b}$ or also $\sqrt{a} \pm \sqrt{b} = \sqrt{a + b \pm 2 \cdot \sqrt{a \cdot b}}$.

AL-KARAJI continued this by defining powers with any natural exponent and also explained the calculation with the reciprocals of the powers. However, he still lacked the special case of the first power law $x^m \cdot x^{-m} = x^0$ and thus the definition $x^0 = 1$.

Unfortunately, an important, probably even more significant manuscript by AL-KARAJI has been lost.

That such a writing existed, we know from a mathematician who was born 100 years after the death of AL-KARAJI and already in his early years wrote a work in which he dealt with AL-KARAJI's treatise: IBN YAHYA AL-MAGRHRIBI AL-SAMAW'AL (1130 – 1180), the fourth important algebraist of the Islamic cultural circle after AL-KHWARIZMI, ABU KAMIL and AL-KARAJI.

AL-SAMAW'AL's father, a Jewish religious and linguistic scholar, had immigrated to Baghdad from Morocco around 1130. His son proved to be inquisitive and studious, and soon he had reached the mathematical proficiency of his teachers. Finding no one in Baghdad to explain the elements of EUCLID to him, he worked through this work and the writings of ABU KAMIL and AL-KARAJI on his own. At the age of 19, he wrote *al-Bahir fi'l-jabr* (The Luminous Book on Arithmetic), in which he frequently quoted AL-KARAJI's lost treatise, so that it is not always clear which parts of the work actually originated from whom. However, in some places AL-SAMAW'AL clearly pointed out the progress he had made in comparison to his great role model AL-KARAJI.

In AL-SAMAW'AL one finds the following table for the justification of the power rules; presumably AL-KARAJI proceeded in the same way. In the column headers, the exponents are written according to today's notation, and below them the powers are written in the verbal form usual at that time. Any positive number considered, in the literal translation is called a "thing" (the German mathematicians of the 15th century used the word *coss*). To indicate higher exponents, the first powers are linked linguistically: The Arabic word *māl* stands for the square of a number, *ka'b* for the 3rd power, then the 4th power is described with *māl māl*, the 5th power with *māl ka'b*, the 6th power with *ka'b ka'b*, etc. For the reciprocals of the powers, the Arabic word for *part* is prefixed. The third row of the table shows an example of how the powers of the number 2 are written down (in the case of fractions, this is sometimes only done in product form):

Exponent	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
Notation	part of *ka'b ka'b*	part of *māl ka'b*	part of *māl māl*	part of *ka'b*	part of *māl*	part of thing	unit	thing	*māl*	*ka'b*	*māl māl*	*māl ka'b*	*ka'b ka'b*
Example	$\frac{1}{8}\cdot\frac{1}{8}$	$\frac{1}{4}\cdot\frac{1}{8}$	$\frac{1}{4}\cdot\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64

For the exponents AL-KARAJI and AL-SAMAW'AL use the term *order*. With this they explain the 1st power law $x^m \cdot x^n = x^{m+n}$ as follows:

- *The distance of the order of the product of the two factors from the order of one of the two factors is equal to the distance of the order of the other factor from the unit. If the factors are in different directions, then we count* [*the distance*] *from the order of the first factor towards the unit, but, if they are in the same direction, we count away from the unit.* (quoted from L. BERGGREN).

From today's point of view, such a description of the power law may seem primitive; however, it represents a significant step of abstraction in the development of algebra.

Consider also the difficulties associated with the introduction of negative numbers and the application of the sign rules. AL-KARAJI states:

- *If we multiply a deficient (negative) number* (*interpreted as a deficit*) *by an excess (positive number), the result is a negative number; if we multiply a negative number by a negative number, the result is a positive number.*

And further:

- *If we subtract an excess number from an empty order (zero), the same number, deficient, remains* (*meaning:* $0-a=-a$). *But if we subtract a deficient number from an empty order (zero), there remains in it that number, excess* (*meaning:* $0-(-a)=a$).

The above-mentioned lost writing of AL-KARAJI must also have contained a scheme from which one could read off the coefficients for the powers of *a, b* which result when calculating the sum powers $(a+b)^3$, $(a-b)^3$, $(a+b)^4$, etc. – many centuries before PASCAL's *arithmetic triangle*.
He justified the addition principle for the coefficients with the help of the calculation rules for polynomials.

1	1	1	1	1	1
1	2	3	4	5	…
	1	3	6	10	…
		1	4	10	…
			1	5	…
				1	…
					1

The derivation procedure is astonishingly modern:
He traces the case *n* = 3 back to the insights at *n* = 2, the case *n* = 4 back to *n* = 3, etc., but without explicitly emphasising this *incomplete induction* as a principle of proof.

AL-KARAJI also went beyond his predecessors with regard to a proof of the sum formulae for powers of natural numbers. The Pythagoreans already knew that the sum $1+2+3+...+n$ could be calculated directly with the help of half the product of the largest summand *n* and the following number *n*+1, i.e. with the help of $\frac{1}{2} \cdot n \cdot (n+1)$.

k	Σ k	k³	Σ k³	Σ k³
0	0	0	0	0
1	1	1	1	1²
2	3	8	9	3²
3	6	27	36	6²
4	10	64	100	10²
5	15	125	225	15²

And in Indian mathematicians of the 5th century, one finds $1^3+2^3+3^3+...+n^3=(1+2+3+...+n)^2$, albeit without proof, but as was customary in this cultural sphere: one recognises the connection by looking (cf. table on the left).

AL-KARAJI carried out the formal proof of the formula for the sum of the cubes again according to the inductive method. First he established that it holds (cf. diagram):

$$(1+2+3+4+5)^2=(1+2+3+4)^2+2\cdot(1+2+3+4)\cdot 5+5^2$$

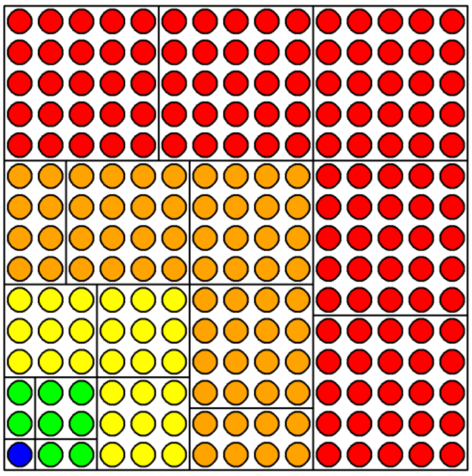
After substituting $1+2+3+4$ by $\frac{1}{2}\cdot 4\cdot 5$ this results in

$$(1+2+3+4+5)^2=(1+2+3+4)^2+4\cdot 5^2+5^2$$

thus $(1+2+3+4+5)^2=(1+2+3+4)^2+5^3$.

Since this relationship can also be shown for partial sums, it follows step by step:

$$(1+2+3+4+5)^2=(1+2+3+4)^2+5^3=(1+2+3)^2+4^3+5^3$$
$$=(1+2)^2+3^3+4^3+5^3=1^2+2^3+3^3+4^3+5^3$$
$$=1^3+2^3+3^3+4^3+5^3$$



(from “Mathematics is beautiful”)

As mentioned above, AL-KARAJI dealt intensively with the *Arithmetica* of DIOPHANTUS. He took up many of its tasks and generalised them and above all, however, he checked to what extent the tricky approaches of DIOPHANTUS could be generalised. For example, to solve the equation $x^3+y^3=u^2$, i.e. the question of which sums of cubic numbers are square numbers, he used the approach $y=m\cdot x$ and $u=n\cdot x$ with $m,\ n\in\mathbb{Q}$, which then led to a parameter representation of the solution triples $x=\frac{n^2}{1+m^3}$, $y=\frac{m\cdot n^2}{1+m^3}$, $u=\frac{n^3}{1+m^3}$.

While AL-KARAJI was "only" able to divide polynomials by monomials, AL-SAMAW'AL even managed the problem of dividing any polynomials. He wrote down the terms of the division $(20x^2+30x)/(6x^2+12)$ in tabular form; the coefficients of the dividend (line 1a) and divisor (line 1b) are placed one below the other (adding zeros to non-existent powers). In line 0, the intermediate results of the division are entered consecutively. The result of the division of the highest powers, i.e. of $3\frac{1}{3}$, is entered in line 0, then the $3\frac{1}{3}$-fold of line 1b is subtracted from line 1a and noted in line 2a.

This process is now repeated *with the divisor shifted by one order in the manner of Indian arithmetic*, the result 5 of the division of 30*x* / 6*x* is entered in the next field of line 0, and so on.

From the table one can even read without further calculation which coefficients result in the next steps. Thus – as AL-SAMAW'AL wrote – the result was an *approximate* solution:

$$(20x^2+30x):(6x^2+12)\approx 3\tfrac{1}{3}+5\cdot x^{-1}-6\tfrac{2}{3}\cdot x^{-2}-10\cdot x^{-3}+13\tfrac{1}{3}\cdot x^{-4}+20\cdot x^{-5}-26\tfrac{2}{3}\cdot x^{-6}-40\cdot x^{-7}$$

Exponent	2	1	0	-1	-2	-3	-4	-5	-6	-7
Line 0			$3\frac{1}{3}$	5	$-6\frac{2}{3}$	-10	$13\frac{1}{3}$	20	$-26\frac{2}{3}$	-40
Line 1a	20	30	0							
Line 1b	6	0	12							
Line 2a		30	-40	0						
Line 2b		6	0	12						
Line 3a			-40	-60	0					
Line 3b			6	0	12					
Line 4a				-60	80	0				
Line 4b				6	0	12				
Line 5a					80	120	0			
Line 5b					6	0	12			
Line 6a						120	-160	0		
Line 6b						6	0	12		
Line 7a							-160	-240	0	
Line 7b							6	0	12	
Line 8a								-240	320	0
Line 8b								6	0	12
									320	480

In another point AL-SAMAW'AL surpassed his model AL-KARAJI: He succeeded not only in making denominators of fractions with two roots as summands rational, but even those with three roots:

$$\frac{\sqrt{30}}{\sqrt{2}+\sqrt{5}+\sqrt{6}} = \frac{-20\cdot\sqrt{2}+2\cdot\sqrt{5}+5\cdot\sqrt{6}+6\cdot\sqrt{15}}{13}.$$

In the last chapter of his work *al-Bahir fi'l-jabr*, AL-SAMAW'AL dealt with a combinatorial problem:

10 unknown quantities are sought; the sums of 6 of each of these quantities is given.
He recognises that it is possible to form sums with 6 variables in 210 ways.
Thus, 210 different equations can be set up; however, the selection of the 10 equations for a system of equations to be solved is not arbitrary.

The polymath AL-SAMAW'AL wrote more than 85 texts on various subjects. He travelled to numerous countries in the Islamic world; he earned his living successfully as a doctor.

In a book on the errors of astrologers, he argued that an astrologer must consider 6817 variables for each person in order to make a meaningful prediction about their fate, which cannot be done. He also grappled with the question of true religion, which eventually led him to convert to Islam (though he tried to conceal this from his father for a long time).

It is said of AL-KARAJI that at some point he decided to go from Baghdad to the mountains to devote himself mainly to the question of how to use underground water supplies and improve water management in the fields.

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