LORENZO MASCHERONI (May 15, 1750 – July 14, 1800)

by HEINZ KLAUS STRICK, Germany

LORENZO MASCHERONI may not have been as famous a mathematician as some of his contemporaries, but two of his publications were so important that his name is inseparably linked with two fields of mathematics.

- The EULER-MASCHERONI constant $\gamma \approx 0.577215\ldots$ is the limit of the sequence $H_n - \ln(n)$ i.e. the sequence by which the partial sums of the harmonic series can be approximately determined with the help of the logarithm of the associated natural number.
- The theorem of MOHR-MASCHERONI states that any construction that can be done with a compass and ruler can even be done with the compass alone.

LORENZO MASCHERONI was born the son of a rich landowner in Bergamo (Lombardy, then part of the Republic of Venice). At the age of 17 he was ordained a priest and at 20 he taught rhetoric at the *Seminario di Bergamo*, and from 1778 onwards mathematics and physics at the *Collegio Mariano*. In 1785 he published a work on statics: *Nuove ricerche su l'equilibrio delle volte* (New research on the equilibrium of vaults), after which he was appointed to the chair of algebra and geometry at the University of Pavia. From 1789 to 1793 he held the office of rector of the university.

In 1790 his *Adnotationes ad calculum integrale Euleri* (Notes on EULER's integrals) appeared. In this treatise MASCHERONI showed that he had mastered the methods developed by LEONHARD EULER. EULER had calculated the constant γ to six digits in 1734, adding another ten digits a year later. MASCHERONI determined 32 decimal places of the constant (however, as it turns out later, see below, the places 20 to 22 were not correct due to a transmission error).

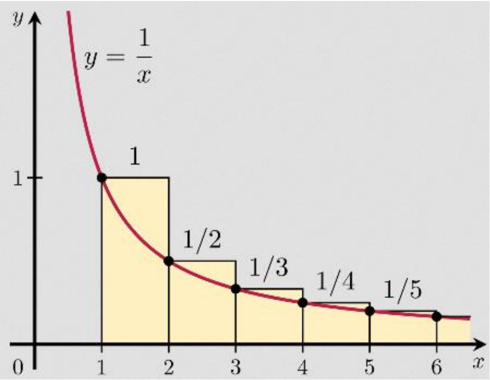
Comparing the graph of the function *f* with $f(x) = \frac{1}{x}$ with the harmonic series

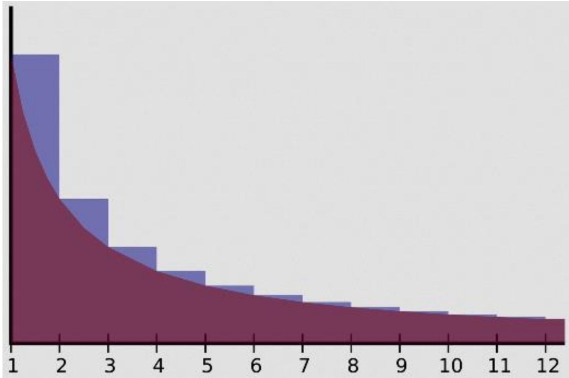
$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n}$, then from the following figure on the left,

because of $\int_1^b \frac{1}{x}\,dx = \ln(b) - \ln(1) = \ln(b)$ one can read off the inequality

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}\,dx < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n} = H_n .$$

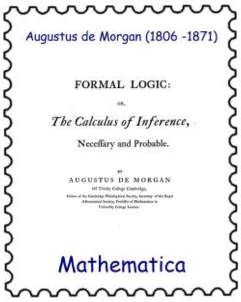
The areas coloured blue in the figure on the right correspond cumulatively to the difference $H_n - \ln(n)$.





Sources: Left: https://commons.wikimedia.org/wiki/File:Integral_Test.svg (Jim Belk)

Right: https://commons.wikimedia.org/wiki/File:Gamma-area.svg (William Demchick: Kiwi128)



EULER had noted the context (in a rather symbolic notation) in 1734:

$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}=\ln(n+1)+C$ with $C=0.577218$.

MASCHERONI himself used the letter *A*. Today's usual designation by "*gamma*" was first made in 1836 by AUGUSTUS DE MORGAN.

Note: Since the difference sequence $\ln(n+1)-\ln(n)$ is a zero sequence, it does not matter whether $H_n-\ln(n)$ or $H_n-\ln(n+1)$ is examined.

For the calculation of the constant, EULER used the power series expansion of the logarithm function:

$$\left(\sum_{r=1}^{n}\frac{1}{r}\right)-\ln(n+1)=\frac{1}{2}\cdot\sum_{r=1}^{n}\frac{1}{r^2}-\frac{1}{3}\cdot\sum_{r=1}^{n}\frac{1}{r^3}+\frac{1}{4}\cdot\sum_{r=1}^{n}\frac{1}{r^4}-+\ldots,$$

whereby he could give limit values for the sums with even powers in each case (cf. solution of the Basel problem):

$$\lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{1}{r^2}\right)=\frac{\pi^2}{6},\ \lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{1}{r^4}\right)=\frac{\pi^4}{90},\ \ldots$$

With the help of error estimates (the so-called EULER-MACLAURIN formula), EULER finally came up with the following representation

$$\gamma=\left(\sum_{r=1}^{n}\frac{1}{r}\right)-\ln(n)-\frac{1}{2n}+\frac{1}{12n^2}-\frac{1}{120n^4}+\frac{1}{252n^6}+-\ldots,$$

which was further developed by MASCHERONI.

In his treatise MASCHERONI also dealt with the so-called *integral logarithm function*, which is given by $Li(x)=\int_2^x\frac{1}{\ln(u)}\,du$.

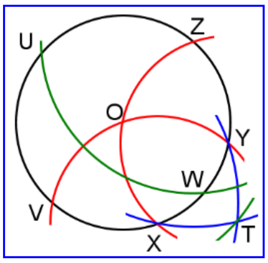
In 1804, the astronomer and mathematician JOHANN GEORG VON SOLDNER, who lived in Berlin, derived the relation

$Li(x)=\gamma+\ln(\ln(x))+\sum_{r=1}^{\infty}\frac{(\ln(x))^n}{r\cdot r!}$ with which he determined γ to 22 digits.

The difference to MASCHERONI's sequence of digits was conspicuous, which in turn prompted CARL FRIEDRICH GAUSS to check the calculation himself and have one of his young students, FRIEDRICH BERNHARD GOTTFRIED NICOLAI, calculate up to the 40th digit.

In 1797, French revolutionary troops under General NAPOLEON conquered Upper Italy and united the conquered territories (the Republic of Venice and Habsburg territories) to form the *Cisalpine Republic*. MASCHERONI, who was in the process of completing his new work *Geometria del Compasso* (Geometry of the Circle), met NAPOLEON and was so impressed by his personality that he spontaneously dedicated the new work to him. The book was printed with a dedication in verse for *Bonaparte Italico* and was immediately translated into French.

NAPOLEON was particularly impressed when MASCHERONI could immediately provide him with a solution to a problem that has gone down in literature as *NAPOLEON's problem*:



- *How can one divide a circle with the compass alone into four equal parts, i.e. inscribe a square on a circle?*

MASCHERONI chose this problem and the construction indicated in the illustration on the right as the introduction to his new book.

NAPOLEON also expressed his enthusiasm to JOSEPH LOUIS LAGRANGE and PIERRE-SIMON LAPLACE.

"*Nous attendions tout de vous, Général, excepté des leçons de Matématiques*." [We would have expected everything from you, General, except lessons in mathematics.] the Interior Minister LAPLACE, newly appointed by NAPOLEON, is said to have exclaimed.

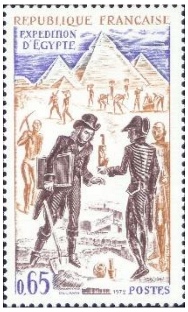
MASCHERONI was not the first to deal with the question of which constructions are feasible with the compass alone. The Danish mathematician GEORG MOHR had already dealt with this in 1672. However, his work *Euclidis Danicus*, which was only rediscovered in 1927, was not widely read in the 17th century and was probably hardly known to MASCHERONI.

MASCHERONI justified his investigations with the fact that all geometric problems depend on points to be constructed. In detail, he explained how to bisect an arc whose centre and radius are given, how to add and subtract lengths by construction, how to find the fourth proportional to three given lengths, how to construct the intersection of two given straight lines and the intersection of a straight line with a circle. With the help of these basic tasks, he showed that the use of the compass was sufficient for all constructions that could be carried out with a compass and ruler.

The book also contained practicable approximate constructions for problems that could not be solved exactly by construction.

In the meantime, MASCHERONI was appointed a member of the *Accademia Reale di Mantova* and the *Società Italiana delle Scienze*, and was elected as a deputy to the legislative assembly in Milan.

The government of the Cisalpine Republic sent him to Paris to work on the introduction of the metric system. 1791 the French National Convention had decided that the new unit of longitude should be equal to the ten millionth part of the length of the meridian passing through Paris between the North Pole and the equator. The French astronomers JEAN-BAPTISTE JOSEPH DELAMBRE and PIERRE MÉCHAIN were commissioned to carry out new surveys along the *Dunkirk-Barcelona route*; others, including MASCHERONI, were also involved in the evaluation.



The commission finished its work in December 1799, but MASCHERONI could not return home afterwards, for in the meantime the Cisalpine Republic had been occupied by Austrian troops of the anti-NAPOLEON coalition.

The Habsburgs had taken advantage of NAPOLEON's absence during his *Egypt campaign* to attack. And so it happened that MASCHERONI – seriously ill after a cold – died in Paris.

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https://www.spektrum.de/wissen/lorenzo-mascheroni/1726824

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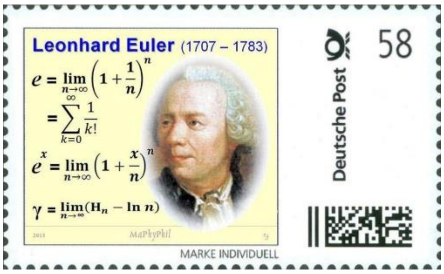

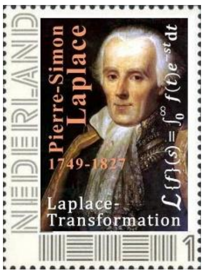

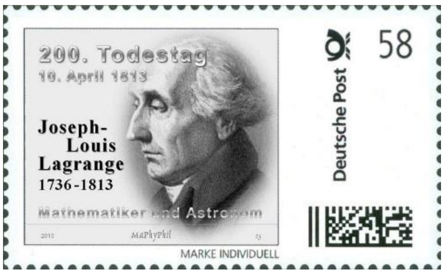

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