Residue Class-Wise Affine Representations of Groups

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Preliminary remarks: Let $\mathbb{K}$ be a category. A $\mathbb{K}$-representation of a monoid G is an homomorphism

$$\varphi : \ G \ \longrightarrow \ \mathrm{End}_{\mathbb{K}}(X)$$

for some object X of $\mathbb{K}$. In representation theory G usually is a group and $\mathbb{K}$ typically is the category of finite-dimensional vector spaces over a field or the category of finite-dimensional free modules over a ring.

We consider the case that $\mathbb{K}$ is the category of infinite principal ideal domains R all of those nontrivial residue class rings are finite, endowed with a topology by taking the set of all residue classes as a basis.

Depending on the base ring we will restrict our attention to continuous mappings of a certain kind.

For purposes of saving (lots of) space we usually do not give proofs here.

Definition 1: Let R be as above, hence e.g. $R = \mathbb{Z}, \mathbb{Z}[i], \mathbb{F}_q[x], \mathbb{Z}_{(\pi)}, \dots$.

We call a mapping $f : R \to R$ *residue class-wise affine* or shortly *rcwa*-mapping if there is an $m_f \in R \setminus \{0\}$ such that the restriction of f to any residue class $r(m_f) \in R/m_fR$ is given by

$$n \longmapsto \frac{a_r \cdot n + b_r}{c_r}$$

for certain coefficients $a_r, b_r, c_r \in R$.

We always assume that m_f is minimal, i.e. that not already a proper divisor of m_f satisfies the given criteria.

We denote the set of all rcwa mappings of the ring R by $\mathrm{Rcwa}(R)$, and set

$$\mathrm{RCWA}(R) \ := \ \mathrm{Rcwa}(R) \cap \mathrm{Sym}(R).$$

$\mathrm{Rcwa}(R)$ is a monoid and $\mathrm{RCWA}(R)$ is a proper subgroup of $\mathrm{Sym}(R)$ – proof: elementary and easy.

Example 1: Although to my knowledge rcwa mappings have not been investigated as such in general before, there is one very popular such mapping:

$$T \in \mathrm{Rcwa}(\mathbb{Z}) : \; n \longmapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ even}, \\ \frac{3n+1}{2} & \text{if } n \text{ odd}. \end{cases}$$

The Collatz conjecture asserts that

$$\forall n \in \mathbb{N} \;\; \exists k \in \mathbb{N} : \quad n^{T^k} = 1.$$

This is an unproven hypothesis, an extensive commented bibliography listing about 100 publications on this question has been compiled by Jeffrey C. Lagarias. As an example we give the sequence starting with 27:

27, 41, 62, 31, 47, 71, 107, 161, 242, 121, 182, 91, 137, 206, 103, 155, 233, 350, 175, 263, 395, 593, 890, 445, 668, 334, 167, 251, 377, 566, 283, 425, 638, 319, 479, 719, 1079, 1619, 2429, 3644, 1822, 911, 1367, 2051, 3077, 4616, 2308, 1154, 577, 866, 433, 650, 325, 488, 244, 122, 61, 92, 46, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1.

Example 2: We can also get permutations: for example, define $\alpha \in \mathrm{RCWA}(\mathbb{Z})$ by

$$n \longmapsto \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \ (2), \\ \frac{3n+1}{4} & \text{if } n \equiv 1 \ (4), \\ \frac{3n-1}{4} & \text{if } n \equiv 3 \ (4). \end{cases}$$

It can be seen easily that α centralizes the involution $n \mapsto -n$. There have been investigations concerning the finiteness of the cycle containing 8, but to my knowledge this question is still open. Fixed points are 0 and ± 1, the known finite cycles are $\pm(2\ 3)$, $\pm(4\ 6\ 9\ 7\ 5)$ and $\pm(44\ 66\ 99\ 74\ 111\ 83\ 62\ 93\ 70\ 105\ 79\ 59)$.

Example 3: Similarly, we can e.g. permute polynomial rings – define $r \in \mathrm{RCWA}(\mathbb{F}_2[x])$ by

$$P \mapsto \begin{cases} \frac{(x^2+x+1)P}{x^2+1} & \text{if } P \equiv 0(x^2+1), \\ \frac{(x^2+x+1)P+x}{x^2+1} & \text{if } P \equiv 1(x^2+1), \\ \frac{(x^2+x+1)P+x^2}{x^2+1} & \text{if } P \equiv x(x^2+1), \\ \frac{(x^2+x+1)P+(x^2+x)}{x^2+1} & \text{otherwise.} \end{cases}$$

The mapping r fixes the degree of any polynomial, hence any cycle is finite. Despite of this the order of r is infinite since there is no upper bound on the cycle lengths.

Definition 2: Let $f \in \mathrm{Rcwa}(R)$ and m_f, a_r, b_r and c_r be as in Definition 1. We define the

- **modulus** $\mathrm{Mod}(f)$ of f by $|m_f|$, the

- **multiplier** $\mathrm{Mult}(f)$ of f by $|\operatorname{lcm}_r a_r|$, the

- **divisor** $\mathrm{Div}(f)$ of f by $|\operatorname{lcm}_r c_r|$, and the

- **prime set** $\mathcal{P}(f)$ of f as the set of prime divisors of $\mathrm{Mod}(f) \cdot \mathrm{Mult}(f) \cdot \mathrm{Div}(f)$. (PID's are unique factorization domains!)

$|x|$: 'standard associate' of $x \in R$.

Lemma 1 (composita of rcwa maps.):
For $f, g \in \mathrm{Rcwa}(R)$ the following hold:

1. $\mathrm{Div}(f) | \mathrm{Mod}(f)$,
2. $\mathrm{Mod}(f \cdot g) | \mathrm{Mod}(f) \cdot \mathrm{Mod}(g)$,
3. $\mathrm{Mult}(f \cdot g) | \mathrm{Mult}(f) \cdot \mathrm{Mult}(g)$,
4. $\mathrm{Div}(f \cdot g) | \mathrm{Div}(f) \cdot \mathrm{Div}(g)$,
5. $\mathcal{P}(f \cdot g) \subseteq \mathcal{P}(f) \cup \mathcal{P}(g)$,
6. $p \in R$ prime $\wedge\ p | \mathrm{Mult}(f) \wedge p \nmid \mathrm{Div}(g)$
   $\Rightarrow p | \mathrm{Mult}(f \cdot g)$,
7. $p \in R$ prime $\wedge\ p | \mathrm{Div}(f) \wedge p \nmid \mathrm{Mult}(g)$
   $\Rightarrow p | \mathrm{Div}(f \cdot g)$,
8. f surjective $\wedge\ p \in R$ prime $\wedge\ p \nmid \mathrm{Mult}(f)$
   $\wedge\ p | \mathrm{Div}(g) \Rightarrow p | \mathrm{Div}(f \cdot g)$, and
9. f surjective $\wedge\ p \in R$ prime $\wedge\ p \nmid \mathrm{Div}(f)$
   $\wedge\ p | \mathrm{Mult}(g) \Rightarrow p | \mathrm{Mult}(f \cdot g)$.

Lemma 2 (inverses of rcwa maps.):
For $\sigma \in \mathrm{RCWA}(R)$ the following hold:

1. $\mathrm{Mod}(\sigma^{-1}) \mid \mathrm{Mult}(\sigma) \cdot \mathrm{Mod}(\sigma)$,
2. $\mathrm{Mult}(\sigma) \mid \mathrm{Mod}(\sigma^{-1})$,
3. $\mathrm{Mult}(\sigma^{-1}) = \mathrm{Div}(\sigma)$,
4. $\mathrm{Div}(\sigma^{-1}) = \mathrm{Mult}(\sigma)$, and
5. $\mathcal{P}(\sigma^{-1}) = \mathcal{P}(\sigma)$.

Lemma 3: The following hold:

1. The image of an rcwa mapping is the union of a finite number of residue classes and a finite subset of R.
2. The preimage of a union of residue classes under an rcwa mapping $f \in \mathrm{Rcwa}(R)$ is a union of residue classes, again – or in other words, f is continuous.
3. The sets R, $\mathrm{Rcwa}(R)$ and $\mathrm{RCWA}(R)$ have equal cardinalities.

Definition 3: Let G be a monoid. Then a *residue class-wise affine* (R-) *representation*, or shortly *rcwa representation*, of G is an homomorphism

$$\varphi : G \to \mathrm{Rcwa}(R).$$

Remark 1: Any finite group G has faithful R-rcwa representations.

Definition 4: Let $f : R \to R$ be an rcwa mapping and $m \in R \backslash \{0\}$. Then we define the *transition graph* $\Gamma_{f,m}$ *of* f *for modulus* m as follows:

- The vertices are the residue classes (mod m).
- There is an edge from $r_1(m)$ to $r_2(m)$ if and only if there is an $n_1 \in r_1(m)$ such that $n_1^f \in r_2(m)$.

$\Gamma_{f,m}$ is a directed graph which may have loops.

Example 4: Let $\beta, \gamma \in \mathrm{RCWA}(\mathbb{Z})$ be given by

$$n \longmapsto \begin{cases} n^\alpha + 3 & \text{if } n \equiv 1 \ (4), \\ n^\alpha & \text{otherwise,} \end{cases} \quad \text{resp.}$$

$$n \longmapsto \begin{cases} n^\alpha + 3 & \text{if } n \equiv -1 \ (4), \\ n^\alpha & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} \varphi: \ \mathrm{S}_{10} &\longrightarrow \mathrm{RCWA}(\mathbb{Z}), \\ (1\ 2\ 3\ 4\ 6\ 8) &\longmapsto [\alpha, \beta], \\ (3\ 5\ 7\ 6\ 9\ 10) &\longmapsto [\alpha, \gamma] \end{aligned}$$

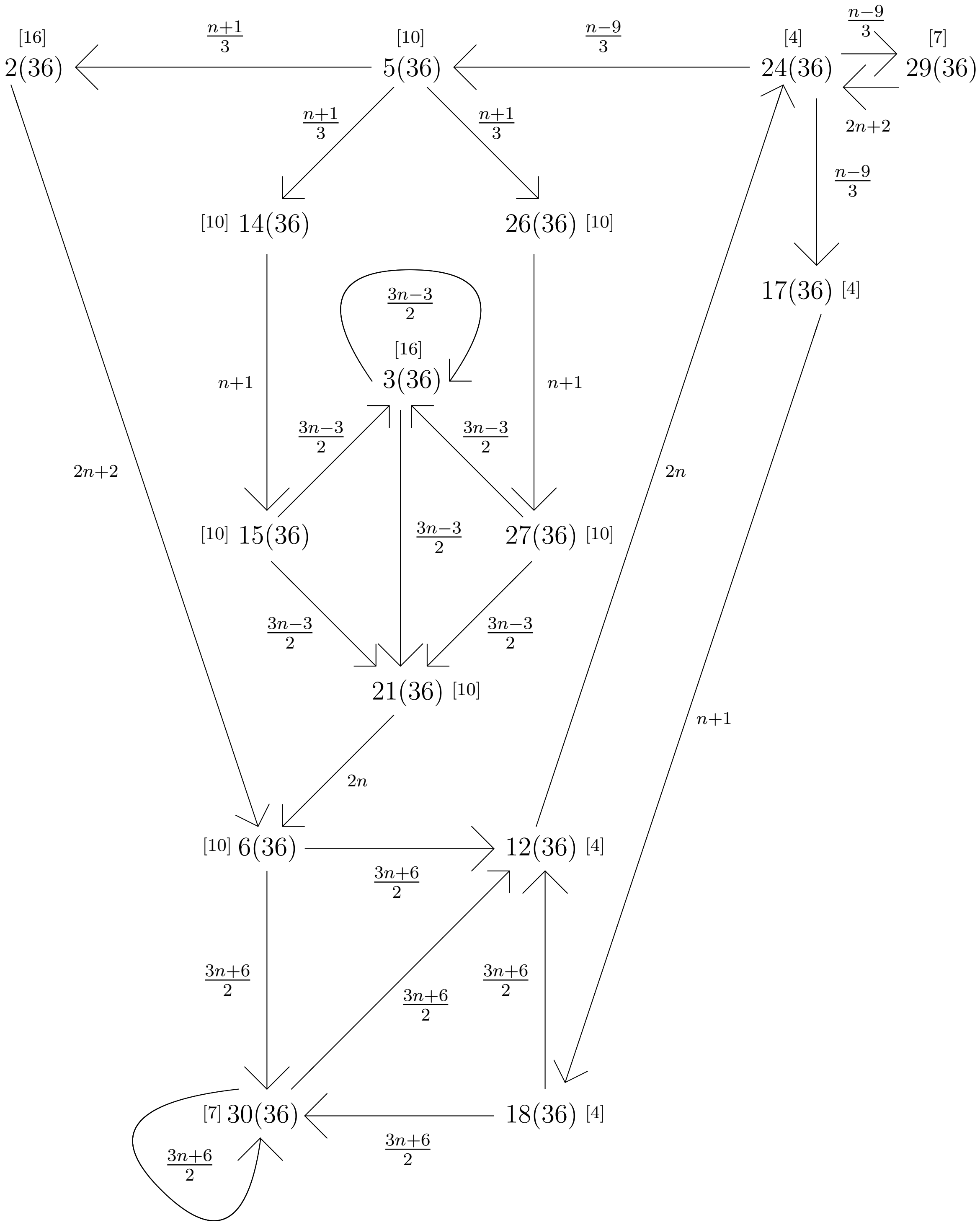
is a faithful rcwa representation of the symmetric group on 10 points. The mapping $[\alpha, \beta]$ is given by

$$n \longmapsto \begin{cases} n & \text{if } n \equiv 0, 2, 3, 8 \ (9), \\ 2n - 5 & \text{if } n \equiv 1 \ (9), \\ n + 3 & \text{if } n \equiv 4, 7 \ (9), \\ 2n - 4 & \text{if } n \equiv 5 \ (9), \\ \frac{n+2}{2} & \text{if } n \equiv 6 \ (18), \\ \frac{n-5}{2} & \text{if } n \equiv 15 \ (18). \end{cases}$$

Example 5: We define $\sigma \in \mathrm{RCWA}(\mathbb{Z})$ by

$$n \longmapsto \begin{cases} \frac{3n-3}{2} & \text{if } n \equiv 3 \ (12), \\ \frac{3n+6}{2} & \text{if } n \equiv 6 \ (12), \\ \frac{n+1}{3} & \text{if } n \equiv 5 \ (36), \\ \frac{n-9}{3} & \text{if } n \equiv 24 \ (36), \\ 2n & \text{if } n \equiv 12, 21 \ (36), \\ 2n+2 & \text{if } n \equiv 2, 29 \ (36), \\ n+1 & \text{if } n \equiv 14, 17, 26 \ (36), \\ n & \text{otherwise.} \end{cases}$$

The mapping σ is a permutation of infinite order, which presumably has only finite cycles (!).



The transition graph $\Gamma_{\sigma,36}$.

Definition 5:

- We say that an rcwa mapping f is *tame* if the set of moduli of its powers is bounded, and *wild* otherwise.
- We say that an rcwa mapping f is *flat* if $\mathrm{Mult}(f) = \mathrm{Div}(f) = 1$, and *balanced* if $\mathrm{Mult}(f)$ and $\mathrm{Div}(f)$ have the same sets of prime divisors. The flat mappings form a submonoid of $\mathrm{Rcwa}(R)$. Obviously, any flat mapping is tame.

Examples 6: The mapping

$$n \longmapsto \begin{cases} n & \text{if } n \equiv 0 \ (7), \\ n+1 & \text{if } n \equiv 1,2,3,4,5 \ (7), \\ n-5 & \text{if } n \equiv 6 \ (7) \end{cases}$$

is flat and conjugate to $[\alpha, \beta]$ as well as $[\alpha, \gamma]$.

The mapping

$$n \longmapsto \begin{cases} 16n+2 & \text{if } n \equiv 0 \ (32), \\ 16n+18 & \text{if } n \equiv 1 \ (2) \\ & \text{and } n \not\equiv -1 \ (32), \\ n-31 & \text{if } n \equiv -1 \ (32), \\ \frac{n}{16} & \text{if } n \equiv 16 \ (32), \\ n+16 & \text{if } n \equiv 2,4,\dots,14 \ (32), \\ n-14 & \text{otherwise} \end{cases}$$

has order 257.

Definition 6: We generalize the terms *modulus*, *multiplier*, *divisor*, *prime set*, *tame*, *wild* and *flat* to rcwa monoids G and rcwa representations in a natural way – let

$$\begin{aligned} \mathrm{Mod}(G) &:= \operatorname*{lcm}_{g \in G} \mathrm{Mod}(g), \\ \mathrm{Mult}(G) &:= \operatorname*{lcm}_{g \in G} \mathrm{Mult}(g), \\ \mathrm{Div}(G) &:= \operatorname*{lcm}_{g \in G} \mathrm{Div}(g) \text{ and} \\ \mathcal{P}(G) &:= \cup_{g \in G} \mathcal{P}(g), \end{aligned}$$

where we set $\mathrm{Mod}(G) := 0$, $\mathrm{Mult}(G) := \infty$ resp. $\mathrm{Div}(G) := \infty$, if the respective lcm's do not exist. Further define

$$\begin{aligned} G \text{ tame} &:\Longleftrightarrow \mathrm{Mod}(G) \neq 0, \\ G \text{ wild} &:\Longleftrightarrow \mathrm{Mod}(G) = 0 \text{ and} \\ G \text{ flat} &:\Longleftrightarrow \mathrm{Mult}(G) = \mathrm{Div}(G) = 1. \end{aligned}$$

Lemma 4: For monoids $G, H \leq \text{Rcwa}(R)$ the following hold:

1. $G \leq \text{RCWA}(R) \Rightarrow \text{Mult}(G) | \text{Mod}(G)$,
2. $\text{Div}(G) | \text{Mod}(G)$,
3. $H \leq G \Rightarrow \text{Mod}(H) | \text{Mod}(G)$,
4. $H \leq G \Rightarrow \mathcal{P}(H) \subseteq \mathcal{P}(G)$, and
5. $G \leq \text{RCWA}(R) \Rightarrow \mathcal{P}(G)$ is the set of prime divisors of $\text{Mod}(G)$.

Here we set $0|0$ and $\infty|0$.

Theorem 1: Tameness is a class invariant: let $f \in \text{Rcwa}(R)$ and $\sigma \in \text{RCWA}(R)$ – if f is tame then so is f^σ.

Proof: We can choose $m \in R \backslash \{0\}$ such that

$\forall n \in \mathbb{Z} \quad \text{Mod}(f^n)|m,$ if f is bijective, resp.

$\forall n \in \mathbb{N} \quad \text{Mod}(f^n)|m$ otherwise.

We have $\text{Mod}((f^\sigma)^n) = \text{Mod}(\sigma^{-1} \cdot f^n \cdot \sigma)$, and by Lemma 1, Assertion (2) this divides $\text{Mod}(\sigma^{-1}) \cdot \text{Mod}(f^n) \cdot \text{Mod}(\sigma)$, hence due to the above also $m \cdot \text{Mod}(\sigma) \cdot \text{Mod}(\sigma^{-1})$. Since the latter expression does not depend on n, we are done. $\square$

Theorem 2: If $f \in \mathrm{Rcwa}(R)$ is surjective but not injective, then f is wild.

Note that in Theorem 2 we do not say anything about the coefficients – surjectivity and non-injectivity are sufficient to conclude that the moduli of the powers of the mapping are not bounded.

Theorem 3: If $f \in \mathrm{Rcwa}(R)$ is surjective but not balanced, then f is wild.

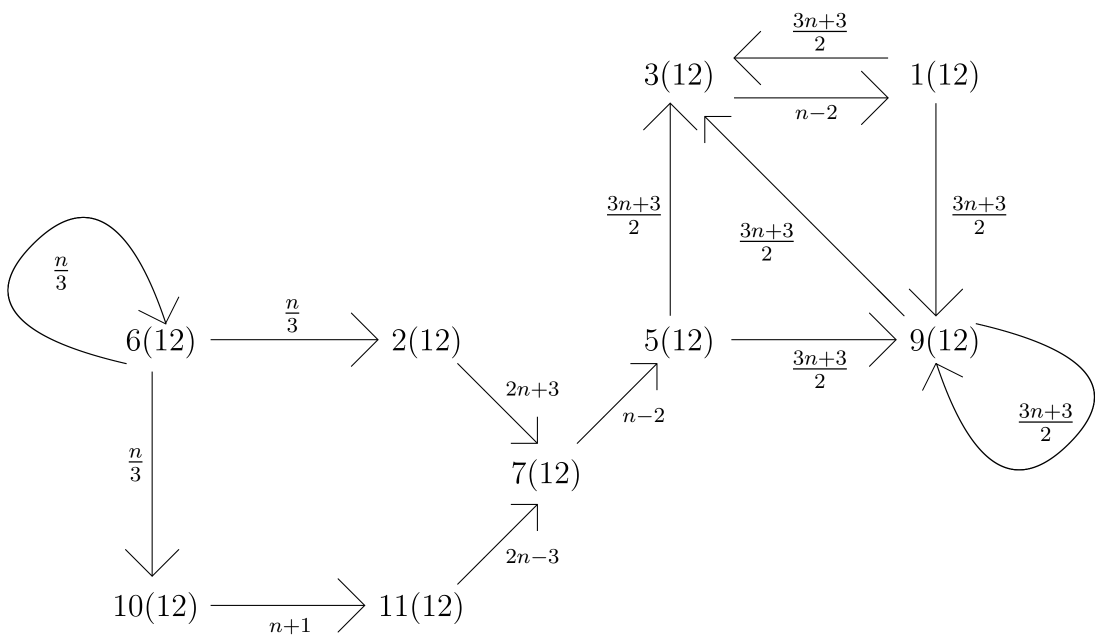
Theorem 4: If $f \in \mathrm{Rcwa}(R)$ and $S \subseteq R$ is a union of finitely many residue classes such that $S^f \supsetneq S$, then f is wild.

Theorem 5: If $f \in \mathrm{Rcwa}(R)$ is surjective and if there is an $m \in R$ such that the transition graph $\Gamma_{f,m}$ of f for module m has weakly-connected components which are not strongly-connected, then f is wild.

Example 7:

$$\sigma \in \mathrm{RCWA}(\mathbb{Z}),$$

$$n \longmapsto \begin{cases} n & \text{if } n \equiv 0 \ (4), \\ \frac{3n+3}{2} & \text{if } n \equiv 1 \ (4), \\ 2n+3 & \text{if } n \equiv 2 \ (12), \\ n-2 & \text{if } n \equiv 3, 7 \ (12), \\ \frac{n}{3} & \text{if } n \equiv 6 \ (12), \\ n+1 & \text{if } n \equiv 10 \ (12), \\ 2n-3 & \text{if } n \equiv 11 \ (12). \end{cases}$$



Definition 7: Let $f \in \mathrm{Rcwa}(R)$ and

$$S_0 \subsetneq S_1 \subsetneq S_2 \subsetneq \dots$$

be an ascending sequence of finite subsets of R such that

1. $S_0^f = S_0$, that
2. for any $k \in \mathbb{N}$ the set S_k is the full preimage of S_{k-1} under f, and that
3. $R = \bigcup_{k=0}^{\infty} S_k$.

Then we call $(S_k)_{k \in \mathbb{N}_0}$ a *contraction sequence* of f. If the mapping f has a contraction sequence then we say that f is *contracting* and call the set S_0 the *contraction centre* of f. Assuming their existence, contraction sequence and contraction centre are determined uniquely.

Remark 2: The property *contracting* is a class invariant.

Example 8: Presumably the Collatz mapping T (see Example 1) is contracting, with contraction centre

$$S_0 = \{ -136, -91, -82, -68, -61, \\ -55, -41, -37, -34, -25, \\ -17, -10, -7, -5, -1, 0, 1, 2\}$$

– proving this would certainly solve the Collatz problem. The sets $S_1, S_2, \ldots, S_{25}$ then would have the cardinalities 30, 42, 66, 95, 138, 187, 258, 345, 467, 627, 848, 1138, 1529, 2041, 2731, 3646, 4865, 6485, 8651, 11529, 15384, 20506, 27312, 36379 resp. 48497.

Example 9: The mapping $T_7 \in \mathrm{Rcwa}(\mathbb{Z})$,

$$n \longmapsto \begin{cases} \frac{n}{6} & \text{if } n \equiv 0 \ (6), \\ \frac{7n+1}{2} & \text{if } n \equiv 1, 5 \ (6), \\ \frac{n}{2} & \text{if } n \equiv 2, 4 \ (6), \\ \frac{n}{3} & \text{if } n \equiv 3 \ (6). \end{cases}$$

is probably also contracting, with contraction centre

$$S_0 = \{-360, -206, -103, -66, -60, -59, -38, -19, -17, -11, -10, -5, -3, -1, 0, 1, 2, 4, 19, 38, 65, 67, 143, 167, 195, 228, 235, 429, 501, 585, 823, 1103, 1287, 2206, 2521, 2881, 3861, 4412, 5042, 8824, 10084\}.$$

This is highly non-trivial, e.g. the first iterate in the trajectory of 9595 under T_7 lying in S_0 is the 4361th one, and the maximum of this sequence is

$$45266766717824274611851780017733940744283387822 72.$$

This maximum is reached at iterate no. 1855.

Definition 8: Let $R \in \{\mathbb{Z}, \mathbb{Z}_{(\pi)}\}$. We call $f \in \mathrm{Rcwa}(R)$ *monotonizable* if there is some $\sigma \in \mathrm{Sym}(R)$ such that f^σ is monotone, and *rcwa-monotonizable* if we can even choose σ to be an rcwa mapping. We call f *nearly (rcwa-) monotonizable* if the above holds at least on $R \backslash S$ for some finite subset S of R.

Lemma 5: If $f \in \mathrm{Rcwa}(\mathbb{Z})$ is surjective, not injective and nearly monotonizable, then f is contracting.

Theorem 6: If $f \in \mathrm{Rcwa}(\mathbb{Z})$ is surjective, not injective and (nearly) rcwa-monotonizable and if $\mathrm{Mult}(f) \neq 0$, then there is some $k \in \mathbb{N}$ such that there are at most finitely many $n \in \mathbb{Z}$ such that $|n^{f^k}| \geq |n|$.

Remark 3: Theorem 6 yields that the Collatz mapping T (see Example 1) is not nearly rcwa-monotonizable, although T is surjective, not injective and we have $\mathrm{Mult}(T) = 3 \neq 0$: if we have $n = 2^k m - 1$ for arbitrary $k, m \in \mathbb{N}$ then

$$n^{T^k} = \frac{3^k n + (3^k - 2^k)}{2^k} > n.$$

Remark 4: In case $R = \mathbb{Z}$ our topology is the one used by Harry Fürstenberg in his topological proof of the infinitude of the set of primes.

Theorem 7: We know by Lemma 3, Assertion (2) that $\mathrm{RCWA}(R)$ is a group of homoeomorphisms. Further, $f \in \mathrm{Rcwa}(R)$ is continuous, and open if $\mathrm{Mult}(f) \neq 0$.

Theorem 8: If the group $G < \mathrm{RCWA}(\mathbb{Z})$ is tame, then their orbits on $\mathbb{Z}$ are closed.

Theorem 9: If R contains an infinite number of prime elements then RCWA(R) is not finitely generated.

Proof: Since for any prime element $p \in R$ there is a mapping $\sigma_p \in \mathrm{RCWA}(R)$ such that $\mathcal{P}(\sigma_p) = \{p\}$ (for example the one mapping $x \in R$ which are divisible by p to $x + p$ and fixing everything else), and since $\mathcal{P}(\sigma)$ is finite for any $\sigma \in \mathrm{RCWA}(R)$, our assertion is a consequence of Lemma 1, Assertion (5) and Lemma 2, Assertion (5). $\square$

Theorem 10: The action of RCWA($\mathbb{Z}$) on $\mathbb{Z}$ is highly transitive.

By Dixon / Mortimer: Permutation Groups, Cor. 7.2A thus a non-trivial normal subgroup of RCWA($\mathbb{Z}$) acts highly transitive on $\mathbb{Z}$, too.

Lemma 6: Any non-trivial normal subgroup $N \lhd \mathrm{RCWA}(R)$ has a flat element $g \neq 1$.

Question: Is RCWA($\mathbb{Z}$) a simple group? What about RCWA(R) for other rings R?

Definition 9: We say that R *has the residue class splitability property* if any residue class of R can be written as a disjoint union of two other residue classes.

Remark 5: If R has the residue class splitability property, we see inductively that any union of k residue classes of R can also be written as a union of $\tilde{k} > k$ residue classes of R.

The ring $\mathbb{Z}$ for example has the residue class splitability property (for example a residue class $a(m)$ can be written as union of $a(2m)$ and $a+m(2m)$), but the rings $\mathbb{Z}_{(\pi)}$ with $2 \notin \pi$ and $\mathbb{F}_q[x]$ for $q \neq 2$ have not.

Theorem 11: If the ring R has the residue class splitability property, then $\mathrm{RCWA}(R)$ acts transitively on the set of unions of finitely many residue classes of R distinct from $\emptyset$ and R itself.

Remark 6: The requirement of the residue class splitability property in Theorem 11 is essential: e.g. for $R = \mathbb{Z}_{(3)}$ the parity of the number of residue classes in a union is invariant under the action of $\mathrm{RCWA}(R)$, which forces the existence of at least two orbits.

Definition 10: Let $G < \mathrm{RCWA}(R)$ and $\mathcal{P}$ be a partition of R into a finite set of residue classes on which G acts naturally as a permutation group. If the restriction of any element of G to any residue class in $\mathcal{P}$ is affine we say that the group G *respects* the partition $\mathcal{P}$.

Lemma 7: Let $G, H < \mathrm{RCWA}(R)$ be rcwa groups, $\mathcal{P}$ be a partition of R respected by G and H and $\sigma \in \mathrm{RCWA}(R)$ be affine on any element of $\mathcal{P}$. Then the following hold:

1. The group $\langle G, H \rangle < \mathrm{RCWA}(R)$ generated by G and H respects $\mathcal{P}$, also.
2. The group G^σ respects the partition $\mathcal{P}^\sigma$.

Theorem 12: A group $G < \mathrm{RCWA}(R)$ is tame if and only if G respects a partition of R into finitely many residue classes.

Remark 7: If $G < \mathrm{RCWA}(R)$ is tame then by Theorem 12, G respects a partition $\mathcal{P}$ of R. If the action of G on R is transitive, then $\mathcal{P}$ is a block system for G. Hence G acts imprimitive, thus at most simply transitive on R. If $\mathrm{RCWA}(R)$ acts highly transitive on R (by Theorem 10 this holds e.g. for $R = \mathbb{Z}$), this implies that a non-trivial tame group cannot be normal in $\mathrm{RCWA}(R)$.

Theorem 13: If the ring R has the residue class splitability property, then any two tame groups $G, H < \mathrm{RCWA}(R)$ have mutually conjugate tame supergroups.

Theorem 14: If the ring R has residue class rings of any finite non-zero cardinality, then precisely those mappings $\sigma \in \mathrm{RCWA}(R)$ and precisely those finitely-generated rcwa groups $G < \mathrm{RCWA}(R)$ are tame which are conjugate to some flat mapping resp. group.

Theorem 15: A group G has a faithful tame R-rcwa representation if and only if there is an $m \in \mathbb{N}$ such that G is isomorphic to some subgroup of the wreath product

$$\mathrm{Aff}(R) \wr \mathrm{S}_m.$$

Corollary 1: Any tame R-rcwa group G has a faithful linear representation over the quotient field of R.

Theorem 16: For even $r \in \mathbb{N}$ the group $\mathrm{RCWA}(\mathbb{Z})$ has an infinite number of conjugacy classes of elements of order r.

Conjecture 1: For odd $r \in \mathbb{N}$ the group $\mathrm{RCWA}(\mathbb{Z})$ has only as many conjugacy classes of elements of order r as there are subsets of the set of divisors of r with least common multiple r.