Motivation

Very little is currently known about highly transitive permutation groups, i.e. those which are k-fold transitive for any k.

The group of residue class-wise affine bijections of $\mathbb{Z}$ belongs to this class.

It has a rich and interesting group theoretical structure. Explicit machine computation in this group is quite feasible – see the GAP-package RCWA .

It acts as a group of homoeomorphisms on $\mathbb{Z}$ endowed with a topology by taking the set of all residue classes as a basis.

One piece of motivation also comes from the famous $3n+1$ conjecture, which has not been treated by means of group theory so far.

Basic Terms

Let R denote an infinite euclidean ring, which has at least one prime ideal and all of whose proper residue class rings are finite.

We call a mapping $f : R \to R$ *residue class-wise affine*, or in short an *rcwa* mapping, if there is an $m \in R \setminus \{0\}$ such that the restrictions of f to the residue classes $r(m) \in R/mR$ are all affine.

This means that for any residue class $r(m)$ there are coefficients $a_{r(m)}, b_{r(m)}, c_{r(m)} \in R$ such that the restriction of the mapping f to the set $r(m) = \{r + km | k \in R\}$ is given by

$$f|_{r(m)} : \ r(m) \to R,$$

$$n \ \mapsto \ \frac{a_{r(m)} \cdot n + b_{r(m)}}{c_{r(m)}} \ .$$

We call m the *modulus* of f. To make this unique, we always choose m multiplicatively minimal.

Examples

Examples of rcwa mappings of $\mathbb{Z}$:

1. $n \mapsto n+1, \quad n \mapsto -n, \quad n \mapsto n+(-1)^n$.

2. The Collatz mapping

$$T : n \ \mapsto \ \begin{cases} \frac{n}{2} & \text{if } \ n \text{ even}, \\ \frac{3n+1}{2} & \text{if } \ n \text{ odd}. \end{cases}$$

   The $3n+1$ conjecture asserts: Iterated application of T to any positive integer eventually ends up with 1.

3. The permutation

$$\alpha : \quad n \ \mapsto \ \begin{cases} \frac{3n}{2} & \text{if } \ n \equiv 0 \bmod 2, \\ \frac{3n+1}{4} & \text{if } \ n \equiv 1 \bmod 4, \\ \frac{3n-1}{4} & \text{if } \ n \equiv 3 \bmod 4. \end{cases}$$

   The cycle structure of α is ‘unknown’.

Aims

The bijective rcwa mappings of the ring R form a group, denoted by RCWA(R).

So far, my main goal was to find out as much as possible about the group RCWA($\mathbb{Z}$) of the residue class-wise affine bijections of the ring of integers and its subgroups.

Results (I)

The group $\mathrm{RCWA}(\mathbb{Z})$

- has $\mathbb{Z}^\times \cong C_2$ as an epimorphic image,
- has a trivial centre,
- has no solvable normal subgroup $\neq 1$,
- is not finitely generated,
- has finite subgroups of any isomorphism type, and
- has only finitely many conjugacy classes of elements of any given odd order, but infinitely many conjugacy classes of elements of any given even order.

Results (II)

The following hold:

- A finite extension $G \trianglerighteq N$ of a subdirect product N of finitely many infinite dihedral groups has always a monomorphic image in RCWA($\mathbb{Z}$).

- The homomorphisms of a given finite group G of odd order into RCWA($\mathbb{Z}$) are parametrized by the non-empty subsets of the set of equivalence classes of transitive permutation representations of G up to inner automorphisms of RCWA($\mathbb{Z}$).

Results (III)

An affine mapping $n \mapsto (an+b)/c$ of $\mathbb{Q}$ is order-preserving if and only if $a > 0$.

We call a residue class-wise affine mapping of $\mathbb{Z}$ *class-wise order-preserving*, if all of its affine partial mappings are order-preserving.

The following holds: The group $(\mathbb{Z}, +)$ is an epimorphic image of the subgroup

$$\mathrm{RCWA}^{+}(\mathbb{Z}) < \mathrm{RCWA}(\mathbb{Z})$$

of all class-wise order-preserving bijective rcwa mappings of $\mathbb{Z}$.

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Most of the results listed so far can easily be generalized to groups $\mathrm{RCWA}(R)$ over euclidean rings R chosen suitably for the particular case.

Methods (I)

Epimorphisms

$$\text{sgn}: \quad \text{RCWA}(\mathbb{Z}) \to \mathbb{Z}^\times$$

('sign') and

$$\det: \quad \text{RCWA}^+(\mathbb{Z}) \to (\mathbb{Z}, +)$$

('determinant') have been constructed explicitly.

In the notation used in the definition of an rcwa mapping, for $\sigma \in \text{RCWA}(\mathbb{Z})$ we have

$$\det(\sigma) \;=\; \frac{1}{m} \sum_{r(m) \in \mathbb{Z}/m\mathbb{Z}} \frac{b_{r(m)}}{|a_{r(m)}|}$$

and

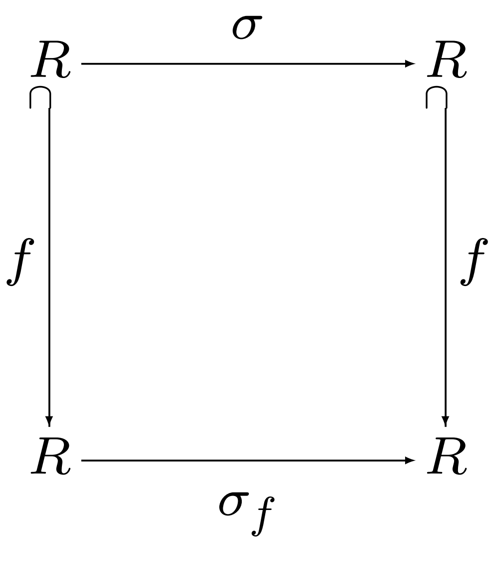
$$\text{sgn}(\sigma) \;=\; (-1)^{\det(\sigma) + \sum_{r(m):\, a_{r(m)}<0} \frac{m-2r}{m}}.$$

Methods (II)

Let $f : R \to R$ be an injective rcwa mapping. Let the *restriction monomorphism*

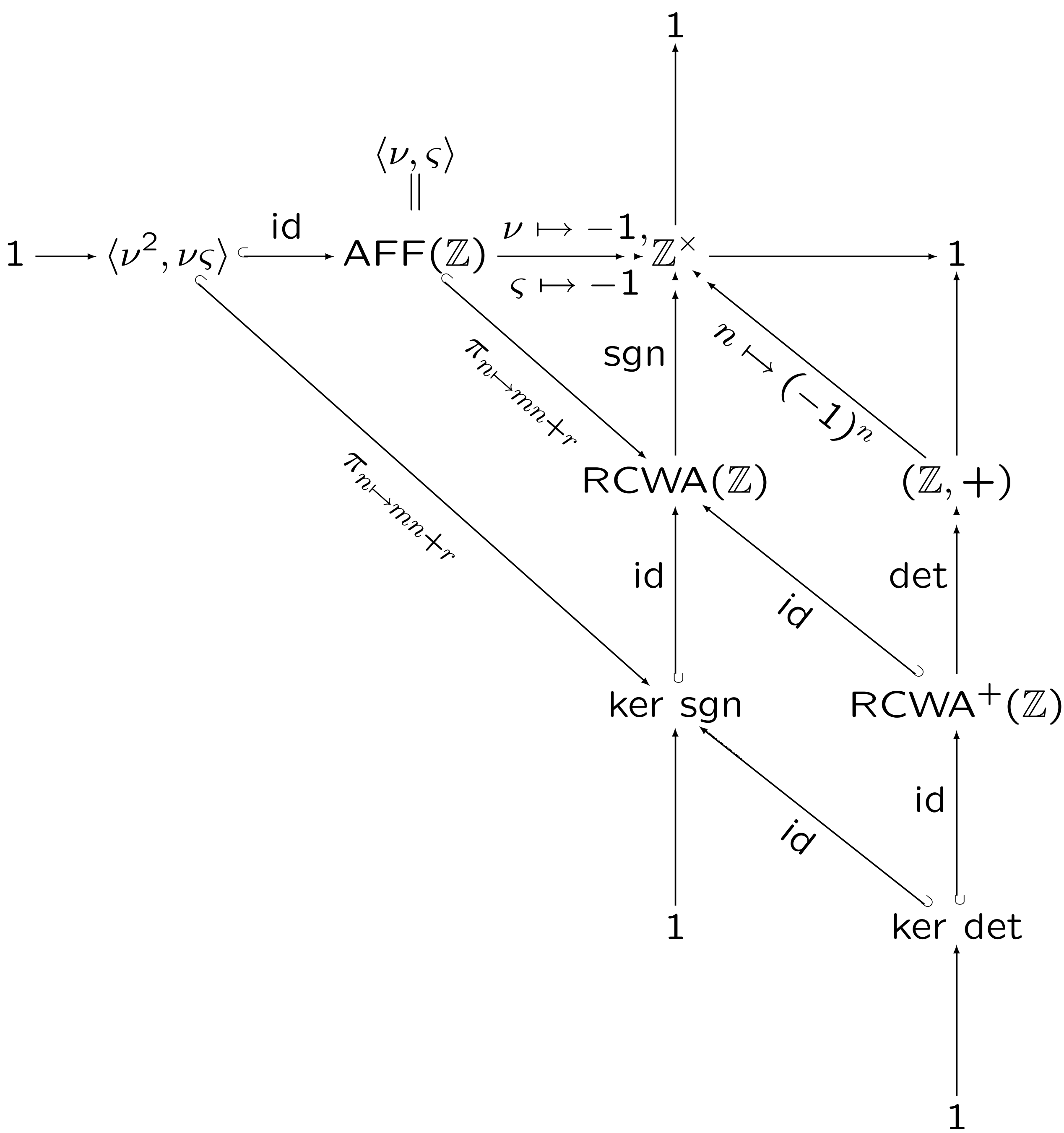
$$\pi_f : \ \mathsf{RCWA}(R) \to \mathsf{RCWA}(R), \quad \sigma \mapsto \sigma_f$$

associated to f be defined such that the diagram



commutes always, and that σ_f always fixes the complement of the image of f pointwise.

Structure



1
⟨ν, ς⟩
1
⟨ν², νς⟩
id
AFF(ℤ)
ν ↦ −1,
ς ↦ −1
ℤ^×
1
sgn
n ↦ (−1)^n
π_{n↦mn+r}
π_{n↦mn+r}
RCWA(ℤ)
(ℤ, +)
id
id
det
ker sgn
RCWA⁺(ℤ)
id
id
1
ker det
1

Example I

The group generated by the permutations

$$\nu : \ n \ \mapsto \ n+1$$

and

$$\tau_{1(2),0(4)} : \ n \ \mapsto \ \begin{cases} 2n-2 & \text{if} \ \ n \equiv 1 \bmod 2, \\ \frac{n+2}{2} & \text{if} \ \ n \equiv 0 \bmod 4, \\ n & \text{if} \ \ n \equiv 2 \bmod 4 \end{cases}$$

acts 3-transitive, but not 4-transitive on $\mathbb{Z}$.

(Proved computationally with RCWA .)

Example II

The group generated by the permutations

$$\alpha: \quad n \ \mapsto \begin{cases} \frac{3n}{2} & \text{if} \ \ n \equiv 0 \text{ mod } 2, \\ \frac{3n+1}{4} & \text{if} \ \ n \equiv 1 \text{ mod } 4, \\ \frac{3n-1}{4} & \text{if} \ \ n \equiv 3 \text{ mod } 4 \end{cases}$$

and

$$\beta: \quad n \ \mapsto \begin{cases} \frac{3n}{5} & \text{if} \ \ n \equiv 0 \text{ mod } 5, \\ \frac{9n+1}{5} & \text{if} \ \ n \equiv 1 \text{ mod } 5, \\ \frac{3n-1}{5} & \text{if} \ \ n \equiv 2 \text{ mod } 5, \\ \frac{9n-2}{5} & \text{if} \ \ n \equiv 3 \text{ mod } 5, \\ \frac{9n+4}{5} & \text{if} \ \ n \equiv 4 \text{ mod } 5 \end{cases}$$

acts (at least!) 2-transitive on the set of positive integers.

(Proved computationally with RCWA .)

Open Questions

- Is $\mathrm{RCWA}(\mathbb{Z}) \rhd \ker \mathrm{sgn} \rhd 1$ a composition series?

- What can be said about the structure of fin.-gen. subgroups of $\mathrm{RCWA}(\mathbb{Z})$? Are they all finitely presented? Can they have intermediate growth?

- Which degrees of transitivity can actions of fin.-gen. subgroups of $\mathrm{RCWA}(\mathbb{Z})$ on $\mathbb{Z}$ or other infinite orbits have?

- Does the group $\mathrm{RCWA}(\mathbb{Z})$ have non-trivial outer automorphisms?

- Find general algorithmic solutions to the membership- / conjugacy problem for fin.-gen. subgroups of $\mathrm{RCWA}(\mathbb{Z})$.

References

RCWA -
[R]esidue [C]lass-[W]ise [A]ffine Groups.
GAP package, 2005.
`www.gap-system.org/Packages/rcwa.html`

Meenaxi Bhattacharjee, Dugald Macpherson, Rögnvaldur G. Möller, and Peter M. Neumann. *Notes on Infinite Permutation Groups*. Number 1698 in Lecture Notes in Mathematics. Springer-Verlag, 1998.

Jeffrey C. Lagarias.
The 3x+1 problem:
An annotated bibliography, 2004.
`arxiv.org/abs/math.NT/0309224`