Convex

A GAP package for handling convex objects.

Version 2015.11.06

August 2012

Sebastian Gutsche

This manual is best viewed as an HTML document. An OFFLINE version should be included in the documentation subfolder of the package.

Sebastian Gutsche
Email: sebastian.gutsche@rwth-aachen.de
Homepage: http://wwwb.math.rwth-aachen.de/~gutsche
Address: Lehrstuhl B für Mathematik, RWTH Aachen, Templergraben 64, 52056 Aachen, Germany
Copyright

© 2011-2012 by Sebastian Gutsche

This package may be distributed under the terms and conditions of the GNU Public License Version 2.

Acknowledgements
## Contents

1 Introduction ................................................. 4
   1.1 What is the goal of the Convex package? .............. 4

2 Installation of the Convex Package ....................... 5

3 Convex Objects .............................................. 6
   3.1 Convex Objects: Category and Representations ........ 6
   3.2 Convex objects: Properties .............................. 6
   3.3 Convex objects: Attributes .............................. 6
   3.4 Convex objects: Methods ............................... 7

4 Fan .......................................................... 8
   4.1 Fan: Category and Representations ....................... 8
   4.2 Fan: Properties ........................................... 8
   4.3 Fan: Attributes ........................................... 9
   4.4 Fan: Methods .............................................. 9
   4.5 Fan: Constructors ........................................ 10
   4.6 Fan: Examples ............................................ 10

5 Cone .......................................................... 11
   5.1 Cone: Category and Representations ..................... 11
   5.2 Cone: Properties .......................................... 11
   5.3 Cone: Attributes .......................................... 11
   5.4 Cone: Methods ............................................. 13
   5.5 Cone: Constructors ....................................... 13
   5.6 Cone: Examples ........................................... 13

6 Polytope ..................................................... 15
   6.1 Polytope: Category and Representations ................. 15
   6.2 Polytope: Properties ..................................... 15
   6.3 Polytope: Attributes .................................... 16
   6.4 Polytope: Methods ...................................... 17
   6.5 Polytope: Constructors ................................... 17
   6.6 Polytope: Examples ..................................... 17
Chapter 1

Introduction

1.1 What is the goal of the Convex package?

Convex provides structures and algorithms for convex geometry. It can handle convex, fans and polytopes. Not only the structures are provided, but also a collection of algorithms to handle those objects. Basically, it provides convex geometry to GAP. It is capable of communicating with the CAS polymake via the package PolymakeInterface and also provides several methods by itself.
Chapter 2

Installation of the **Convex Package**

To install this package just extract the package’s archive file to the GAP pkg directory.

By default the Convex package is not automatically loaded by GAP when it is installed. You must load the package with

```
LoadPackage( "Convex" );
```

before its functions become available.

Please, send me an e-mail if you have any questions, remarks, suggestions, etc. concerning this package. Also, I would be pleased to hear about applications of this package and about any suggestions for new methods to add to the package.

Sebastian Gutsche
Chapter 3

Convex Objects

Convex objects are the main structure of Convex. All other structures, namely fans, cones, and polytopes are derived from this structure. So all methods of this structure also apply to the other data types.

3.1 Convex Objects: Category and Representations

3.1.1 IsConvexObject

\[ \text{IsConvexObject}(M) \]

Returns: true or false

The GAP category of convex objects, the main category of this package.

3.2 Convex objects: Properties

3.2.1 IsFullDimensional

\[ \text{IsFullDimensional}(\text{conv}) \]

Returns: true or false

Checks if the combinatorial dimension of the convex object \( \text{conv} \) is the same as the dimension of the ambient space.

3.3 Convex objects: Attributes

3.3.1 Dimension

\[ \text{Dimension}(\text{conv}) \]

Returns: an integer

Returns the combinatorial dimension of the convex object \( \text{conv} \). This is the dimension of the smallest space in which \( \text{conv} \) can be embedded.

3.3.2 AmbientSpaceDimension

\[ \text{AmbientSpaceDimension}(\text{conv}) \]

Returns: an integer
Returns the dimension of the ambient space of the object \textit{conv}.

3.3.3 ContainingGrid

\begin{itemize}
\item \textbf{ContainingGrid}(\textit{conv}) \hspace{1cm} (attribute)
\item \textbf{Returns:} a homalg module
\item Returns the ambient space of the object \textit{conv} as a homalg module.
\end{itemize}

3.4 Convex objects: Methods

3.4.1 DrawObject

\begin{itemize}
\item \textbf{DrawObject}(\textit{conv}) \hspace{1cm} (operation)
\item \textbf{Returns:} 0
\item Draws a nice picture of the object \textit{conv}, if your computer supports Java. As a side effect, you might not be able to exit \texttt{GAP} anymore.
\end{itemize}

3.4.2 WeakPointerToExternalObject

\begin{itemize}
\item \textbf{WeakPointerToExternalObject}(\textit{conv}) \hspace{1cm} (operation)
\item \textbf{Returns:} a pointer
\item Returns a pointer to an external object which is the basis of \textit{conv}. This method is not used anymore.
\end{itemize}
Chapter 4

Fan

4.1 Fan: Category and Representations

4.1.1 IsFan

▷ IsFan(M) (Category)

Returns: true or false

The GAP category of a fan. Every fan is a convex object.

Remember: Every fan is a convex object.

4.2 Fan: Properties

4.2.1 IsComplete

▷ IsComplete(fan) (property)

Returns: true or false

Checks if the fan fan is complete, i.e. if it’s support is the whole space.

4.2.2 IsPointed

▷ IsPointed(fan) (property)

Returns: true or false

Checks if the fan fan is pointed, which means that every cone it contains is strictly convex.

4.2.3 IsSmooth

▷ IsSmooth(fan) (property)

Returns: true or false

Checks if the fan fan is smooth, i.e. if every cone in the fan is smooth.

4.2.4 IsRegularFan

▷ IsRegularFan(fan) (property)

Returns: true or false

Checks if the fan fan is regular, i.e. if it is the normal fan of a polytope.
4.2.5 IsSimplicial (for a fan)

▷ IsSimplicial(fan) (property)
  Returns: true or false
  Checks if the fan fan is simplicial, i.e. if every cone in the fan is simplicial.

4.2.6 HasConvexSupport

▷ HasConvexSupport(fan) (property)
  Returns: true or false
  Checks if the fan fan is simplicial, i.e. if every cone in the fan is simplicial.

4.3 Fan: Attributes

4.3.1 Rays

▷ Rays(fan) (attribute)
  Returns: a list
  Returns the rays of the fan fan as a list of cones.

4.3.2 RayGenerators

▷ RayGenerators(fan) (attribute)
  Returns: a list
  Returns the generators rays of the fan fan as a list of list of integers.

4.3.3 RaysInMaximalCones

▷ RaysInMaximalCones(fan) (attribute)
  Returns: a list
  Returns a list of lists, which represent an incidence matrix for the correspondence of the rays and the maximal cones of the fan fan. The ith list in the result represents the ith maximal cone of fan. In such a list, the jth entry is 1 if the jth ray is in the cone, 0 otherwise.

4.3.4 MaximalCones

▷ MaximalCones(fan) (attribute)
  Returns: a list
  Returns the maximal cones of the fan fan as a list of cones.

4.4 Fan: Methods

4.4.1 * (for fans)

▷ *(fan1, fan2) (operation)
  Returns: a fan
  Returns the product of the fans fan1 and fan2.
4.5 Fan: Constructors

4.5.1 Fan (For Fans)

- Fan(fan)
  - Returns: a fan
  - Copy constructor for fans. For completeness reasons.

4.5.2 Fan (For a list of rays and a list of cones)

- Fan(rays, cones)
  - Returns: a fan
  - Constructs the fan out of the given rays and a list of cones given by a lists of numbers of rays.

4.6 Fan: Examples

4.6.1 Fan example

```
gap> F := Fan([ [-1,5],[-1,5],[0,1],[1,0],[0,-1]],[[1,2],[2,3],[3,4],[4,1]] );
<A fan in |R^2>
gap> RayGenerators( F );
[ [ -1, 5 ], [ 0, 1 ], [ 1, 0 ], [ 0, -1 ] ]
gap> RaysInMaximalCones( F );
[ [ 1, 1, 0, 0 ], [ 0, 1, 1, 0 ], [ 0, 0, 1, 1 ], [ 1, 0, 0, 1 ] ]
gap> IsRegularFan( F );
true
gap> IsComplete( F );
true
gap> IsSmooth( F );
true
gap> F1 := MaximalCone( F )[ 1 ];
<A cone in |R^2>
gap> DualCone( F1 );
<A cone in |R^2>
gap> RayGenerators( F1 );
[ [ -1, 5 ], [ 0, 1 ] ]
gap> F2 := StarSubdivisionOfIthMaximalCone( F, 1 );
<A fan in |R^2>
gap> IsSmooth( F2 );
true
gap> RayGenerators( F2 );
[ [ -1, 5 ], [ -1, 6 ], [ 0, -1 ], [ 0, 1 ], [ 1, 0 ] ]
```
Chapter 5

Cone

5.1 Cone: Category and Representations

5.1.1 IsCone

\[ \text{IsCone}(M) \] (Category)

\textbf{Returns:} true or false

The GAP category of a cone.

Remember: Every cone is a convex object.

5.2 Cone: Properties

5.2.1 IsRay

\[ \text{IsRay}(\text{cone}) \] (property)

\textbf{Returns:} true or false

Checks if the cone \text{cone} is a ray, i.e. if it has only one ray generator.

5.3 Cone: Attributes

5.3.1 DualCone

\[ \text{DualCone}(\text{cone}) \] (attribute)

\textbf{Returns:} a cone

Returns the dual cone of the cone \text{cone}.

5.3.2 HilbertBasis

\[ \text{HilbertBasis}(\text{cone}) \] (attribute)

\textbf{Returns:} a list

Returns a Hilbert Basis of the cone \text{cone}.
5.3.3 RaysInFacets

\[ \text{RaysInFacets}(\text{cone}) \]

Returns: a list

Returns an incidence matrix for the rays in the facets of the cone \text{cone}. The \( i \)th entry of the result corresponds to the \( i \)th facet, the \( j \)th entry of this is 1 if the \( j \)th ray is in the \( i \)th facet, 0 otherwise.

5.3.4 Facets

\[ \text{Facets}(\text{cone}) \]

Returns: a list

Returns a list of the facets of the cone \text{cone} as homalg cones.

5.3.5 GridGeneratedByCone

\[ \text{GridGeneratedByCone}(\text{cone}) \]

Returns: a homalg module

Returns the grid generated by the lattice points of the cone \text{cone} as a homalg module.

5.3.6 FactorGrid

\[ \text{FactorGrid}(\text{cone}) \]

Returns: a homalg module

Returns the factor of the containing grid of the cone \text{cone} and the grid generated by \text{cone}.

5.3.7 GridGeneratedByOrthogonalCone

\[ \text{GridGeneratedByOrthogonalCone}(\text{cone}) \]

Returns: a homalg module

Returns the grid generated by the lattice points of the orthogonal cone of the cone \text{cone}.

5.3.8 DefiningInequalities

\[ \text{DefiningInequalities}(\text{cone}) \]

Returns: a list

Returns a list of the defining inequalities of the cone \text{cone}.

5.3.9 IsContainedInFan

\[ \text{IsContainedInFan}(\text{cone}) \]

Returns: a fan

If the cone \text{cone} is constructed as part of a fan, this method returns the fan.

5.3.10 FactorGridMorphism

\[ \text{FactorGridMorphism}(\text{cone}) \]

Returns: a morphism

Returns the morphism to the factor grid of the cone \text{cone}.
5.4 Cone: Methods

5.4.1 IntersectionOfCones

\[ \text{IntersectionOfCones(cone1, cone2)} \]

Returns: a cone

If the cones \text{cone1} and \text{cone2} share a face, the method returns their intersection.

5.4.2 Contains

\[ \text{Contains(cone1, cone2)} \]

Returns: true or false

Returns true if the cone \text{cone1} contains the cone \text{cone2}, false otherwise.

5.4.3 StarFan (for a cone)

\[ \text{StarFan(cone)} \]

Returns: a fan

Returns the star fan of the cone \text{cone}, as described in cox, 3.2.7

5.4.4 StarFan (for a cone and a fan)

\[ \text{StarFan(cone, fan)} \]

Returns: a fan

Returns the star fan of the fan \text{fan} along the cone \text{cone}, as described in cox, 3.2.7

5.4.5 StarSubdivisionOfIthMaximalCone

\[ \text{StarSubdivisionOfIthMaximalCone(fan, numb)} \]

Returns: a fan

Returns the star subdivision of the fan \text{fan} on the \text{numb}th maximal cone as in cox, 3.3.13.

5.5 Cone: Constructors

5.5.1 Cone (for a ray list)

\[ \text{Cone(cone)} \]

Returns: a cone

Returns a cone generated by the rays in \text{cone}.

5.6 Cone: Examples

5.6.1 Cone example

Example

\begin{verbatim}
gap> C := Cone([[1,2,3],[2,1,1],[1,0,0],[0,1,1]]);  
<A cone in |R^3>  
gap> Length( RayGenerators( C ) );  
3  
gap> IsSmooth( C );
\end{verbatim}
true

\texttt{gap> Length( HilbertBasis( C ) );
3}

\texttt{gap> IsSimplicial( C );
true}

\texttt{gap> DC := DualCone( C );
<A cone in |R^3>

\texttt{gap> Length( HilbertBasis( DC ) );
3}
Chapter 6

Polytope

6.1 Polytope: Category and Representations

6.1.1 IsPolytope

\[ \text{IsPolytope}(M) \]  
\text{Returns: true or false} 
The GAP category of a polytope. Every polytope is a convex object. Remember: Every cone is a convex object.

6.2 Polytope: Properties

6.2.1 IsNotEmpty

\[ \text{IsNotEmpty}(poly) \]  
\text{Returns: true or false} 
Checks if the polytope \textit{poly} is not empty.

6.2.2 IsLatticePolytope

\[ \text{IsLatticePolytope}(poly) \]  
\text{Returns: true or false} 
Checks if the polytope \textit{poly} is a lattice polytope, i.e. all its vertices are lattice points.

6.2.3 IsVeryAmple

\[ \text{IsVeryAmple}(poly) \]  
\text{Returns: true or false} 
Checks if the polytope \textit{poly} is very ample.

6.2.4 IsNormalPolytope

\[ \text{IsNormalPolytope}(poly) \]  
\text{Returns: true or false} 
Checks if the polytope \textit{poly} is normal.
6.2.5 IsSimplicial (for a polytope)

\( \text{IsSimplicial}(\text{poly}) \)

**Returns:** true or false

Checks if the polytope \( \text{poly} \) is simplicial.

6.2.6 IsSimplePolytope

\( \text{IsSimplePolytope}(\text{poly}) \)

**Returns:** true or false

Checks if the polytope \( \text{poly} \) is simple.

6.3 Polytope: Attributes

6.3.1 Vertices

\( \text{Vertices}(\text{poly}) \)

**Returns:** a list

Returns the vertices of the polytope \( \text{poly} \). For reasons, the corresponding tester is HasVerticesOfPolytopes.

6.3.2 LatticePoints

\( \text{LatticePoints}(\text{poly}) \)

**Returns:** a list

Returns the lattice points of the polytope \( \text{poly} \).

6.3.3 FacetInequalities

\( \text{FacetInequalities}(\text{poly}) \)

**Returns:** a list

Returns the facet inequalities for the polytope \( \text{poly} \).

6.3.4 VerticesInFacets

\( \text{VerticesInFacets}(\text{poly}) \)

**Returns:** a list

Returns the incidence matrix of vertices and facets of the polytope \( \text{poly} \).

6.3.5 AffineCone

\( \text{AffineCone}(\text{poly}) \)

**Returns:** a cone

Returns the affine cone of the polytope \( \text{poly} \).
6.3.6 NormalFan

\[
\text{NormalFan}(\text{poly})
\]

\text{Returns:} \text{ a fan}

Returns the normal fan of the polytope \text{poly}.

6.3.7 RelativeInteriorLatticePoints

\[
\text{RelativeInteriorLatticePoints}(\text{poly})
\]

\text{Returns:} \text{ a list}

Returns the lattice points in the relative interior of the polytope \text{poly}.

6.4 Polytope: Methods

6.4.1 * (for polytopes)

\[
\text{*}(\text{polytope1, polytope2})
\]

\text{Returns:} \text{ a polytope}

Returns the Cartesian product of the polytopes \text{polytope1} and \text{polytope2}.

6.4.2 #

\[
\text{#}(\text{polytope1, polytope2})
\]

\text{Returns:} \text{ a polytope}

Returns the Minkowski sum of the polytopes \text{polytope1} and \text{polytope2}.

6.5 Polytope: Constructors

6.5.1 Polytope (for lists of points)

\[
\text{Polytope}(\text{points})
\]

\text{Returns:} \text{ a polytope}

Returns a polytope that is the convex hull of the points \text{points}.

6.5.2 PolytopeByInequalities

\[
\text{PolytopeByInequalities}(\text{ineqs})
\]

\text{Returns:} \text{ a polytope}

Returns a polytope defined by the inequalities \text{ineqs}.

6.6 Polytope: Examples

6.6.1 Polytope example

\begin{verbatim}
gap> P := Polytope([ [ 2, 0 ], [ 0, 2 ], [ -1, -1 ] ]);  
<A polytope in \mathbb{R}^2>
gap> IsVeryAmple(P);  
true
\end{verbatim}
```gap
LatticePoints( P );
[ [ -1, -1 ], [ 0, 0 ], [ 0, 1 ],
  [ 0, 2 ], [ 1, 0 ], [ 1, 1 ], [ 2, 0 ] ]
NFP := NormalFan( P );
<A complete fan in \( \mathbb{R}^2 \)>
C1 := MaximalCones( NFP )[ 1 ];
<A cone in \( \mathbb{R}^2 \)>
RayGenerators( C1 );
[ [ -1, -1 ], [ -1, 3 ] ]
IsRegularFan( NFP );
true
```
Index

#. 17
* for fans, 9
for polytopes, 17
Convex, 4

AffineCone, 16
AmbientSpaceDimension, 6

Cone for a ray list, 13
ContainingGrid, 7
Contains, 13

DefiningInequalities, 12
Dimension, 6
DrawObject, 7
DualCone, 11

FacetInequalities, 16
Facets, 12
FactorGrid, 12
FactorGridMorphism, 12

Fan For a list of rays and a list of cones, 10
For Fans, 10

GridGeneratedByCone, 12
GridGeneratedByOrthogonalCone, 12

HasConvexSupport, 9
HilbertBasis, 11

IntersectionOfCones, 13
IsComplete, 8
IsCone, 11
IsContainedInFan, 12
IsConvexObject, 6
IsFan, 8
IsFullDimensional, 6
IsLatticePolytope, 15

IsNormalPolytope, 15
IsNotEmpty, 15
IsPointed, 8
IsPolytope, 15
IsRay, 11
IsRegularFan, 8
IsSimplePolytope, 16
IsSimplicial for a fan, 9
for a polytope, 16
IsSmooth, 8
IsVeryAmple, 15

LatticePoints, 16
MaximalCones, 9

NormalFan, 17

Polytope for lists of points, 17
PolytopeByInequalities, 17

RayGenerators, 9
Rays, 9
RaysInFacets, 12
RaysInMaximalCones, 9
RelativeInteriorLatticePoints, 17

StarFan for a cone, 13
for a cone and a fan, 13
StarSubdivisionOfIthMaximalCone, 13

Vertices, 16
VerticesInFacets, 16

WeakPointerToExternalObject, 7