Convex

A GAP package for handling convex objects.

Version 2012.11.15

August 2012

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This manual is best viewed as an HTML document. An OFFLINE version should be included in the documentation subfolder of the package.
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Chapter 1

Introduction

1.1 What is the goal of the Convex package?

Convex provides structures and algorithms for convex geometry. It can handle convex, fans and polytopes. Not only the structures are provided, but also a collection of algorithms to handle those objects. Basically, it provides convex geometry to GAP. It is capable of communicating with the CAS polymake via the package PolymakeInterface and also provides several methods by itself.
Chapter 2

Installation of the Convex Package

To install this package just extract the package’s archive file to the GAP pkg directory.

By default the Convex package is not automatically loaded by GAP when it is installed. You must load the package with

\texttt{LoadPackage( "Convex" );}

before its functions become available.

Please, send me an e-mail if you have any questions, remarks, suggestions, etc. concerning this package. Also, I would be pleased to hear about applications of this package and about any suggestions for new methods to add to the package.

Sebastian Gutsche
Chapter 3

Convex Objects

Convex objects are the main structure of Convex. All other structures, namely fans, cones, and polytopes are derived from this structure. So all methods of this structure also apply to the other data types.

3.1 Convex Objects: Category and Representations

3.1.1 IsConvexObject

\[ \text{IsConvexObject}(M) \]  
\text{(Category)}

Returns: true or false

The GAP category of convex objects, the main category of this package.

3.2 Convex objects: Properties

3.2.1 IsFullDimensional

\[ \text{IsFullDimensional}(conv) \]  
\text{(property)}

Returns: true or false

Checks if the combinatorial dimension of the convex object \(conv\) is the same as the dimension of the ambient space.

3.3 Convex objects: Attributes

3.3.1 Dimension

\[ \text{Dimension}(conv) \]  
\text{(attribute)}

Returns: an integer

Returns the combinatorial dimension of the convex object \(conv\). This is the dimension of the smallest space in which \(conv\) can be embedded.

3.3.2 AmbientSpaceDimension

\[ \text{AmbientSpaceDimension}(conv) \]  
\text{(attribute)}

Returns: an integer
3.3.3 ContainingGrid

\texttt{ContainingGrid(\text{conv})} \hspace{1cm} \text{(attribute)}

\textbf{Returns:} a homalg module

Returns the ambient space of the object \text{conv} as a homalg module.

3.4 Convex objects: Methods

3.4.1 DrawObject

\texttt{DrawObject(\text{conv})} \hspace{1cm} \text{(operation)}

\textbf{Returns:} 0

Draws a nice picture of the object \text{conv}, if your computer supports Java. As a side effect, you might not be able to exit GAP anymore.

3.4.2 WeakPointerToExternalObject

\texttt{WeakPointerToExternalObject(\text{conv})} \hspace{1cm} \text{(operation)}

\textbf{Returns:} a pointer

Returns a pointer to an external object which is the basis of \text{conv}. This method is not used any more.
Chapter 4

Fan

4.1 Fan: Category and Representations

4.1.1 IsFan

\[
\text{IsFan}(\mathcal{M}) \quad \text{(Category)}
\]

Returns: true or false
The GAP category of a fan. Every fan is a convex object.
Remember: Every fan is a convex object.

4.2 Fan: Properties

4.2.1 IsComplete

\[
\text{IsComplete}(\text{fan}) \quad \text{(property)}
\]

Returns: true or false
Checks if the fan \text{fan} is complete, i.e. if it’s support is the whole space.

4.2.2 IsPointed

\[
\text{IsPointed}(\text{fan}) \quad \text{(property)}
\]

Returns: true or false
Checks if the fan \text{fan} is pointed, which means that every cone it contains is strictly convex.

4.2.3 IsSmooth

\[
\text{IsSmooth}(\text{fan}) \quad \text{(property)}
\]

Returns: true or false
Checks if the fan \text{fan} is smooth, i.e. if every cone in the fan is smooth.

4.2.4 IsRegularFan

\[
\text{IsRegularFan}(\text{fan}) \quad \text{(property)}
\]

Returns: true or false
Checks if the fan \text{fan} is regular, i.e. if it is the normal fan of a polytope.
4.2.5 IsSimplicial (for a fan)

\[\text{IsSimplicial}(\text{fan})\]

\textbf{Returns: true or false}

Checks if the fan \textit{fan} is simplicial, i.e. if every cone in the fan is simplicial.

4.2.6 HasConvexSupport

\[\text{HasConvexSupport}(\text{fan})\]

\textbf{Returns: true or false}

Checks if the fan \textit{fan} is simplicial, i.e. if every cone in the fan is simplicial.

4.3 Fan: Attributes

4.3.1 Rays

\[\text{Rays}(\text{fan})\]

\textbf{Returns: a list}

Returns the rays of the fan \textit{fan} as a list of cones.

4.3.2 RayGenerators

\[\text{RayGenerators}(\text{fan})\]

\textbf{Returns: a list}

Returns the generators rays of the fan \textit{fan} as a list of list of integers.

4.3.3 RaysInMaximalCones

\[\text{RaysInMaximalCones}(\text{fan})\]

\textbf{Returns: a list}

Returns a list of lists, which represent an incidence matrix for the correspondence of the rays and the maximal cones of the fan \textit{fan}. The \textit{i}th list in the result represents the \textit{i}th maximal cone of \textit{fan}. In such a list, the \textit{j}th entry is 1 if the \textit{j}th ray is in the cone, 0 otherwise.

4.3.4 MaximalCones

\[\text{MaximalCones}(\text{fan})\]

\textbf{Returns: a list}

Returns the maximal cones of the fan \textit{fan} as a list of cones.

4.4 Fan: Methods

4.4.1 \texttt{* (for fans)}

\[\text{*(fan1, fan2)}\]

\textbf{Returns: a fan}

Returns the product of the fans \textit{fan1} and \textit{fan2}. 
4.5 Fan: Constructors

4.5.1 Fan (For Fans)

\( \triangleright \text{Fan}(\text{fan}) \)  
\textbf{Returns:} a fan  
Copy constructor for fans. For completeness reasons.

4.5.2 Fan (For a list of rays and a list of cones)

\( \triangleright \text{Fan}(\text{rays}, \text{cones}) \)  
\textbf{Returns:} a fan  
Constructs the fan out of the given rays and a list of cones given by a list of numbers of rays.

4.6 Fan: Examples

4.6.1 Fan example

```
gap> F := Fan( [[-1,5],[0,1],[1,0],[0,-1]],[[1,2],[2,3],[3,4],[4,1]] );
<A fan in \( \mathbb{R}^2 \rangle

gap> RayGenerators( F );
[ [ -1, 5 ], [ 0, 1 ], [ 1, 0 ], [ 0, -1 ] ]
gap> RaysInMaximalCones( F );
[ [ 1, 1, 0, 0 ], [ 0, 1, 1, 0 ], [ 0, 0, 1, 1 ], [ 1, 0, 0, 1 ] ]
gap> IsRegularFan( F );
true

gap> IsComplete( F );
true

gap> IsSmooth( F );
true

gap> F1 := MaximalCones( F )[ 1 ];
<A cone in \( \mathbb{R}^2 \rangle

gap> DualCone( F1 );
<A cone in \( \mathbb{R}^2 \rangle

gap> RayGenerators( F1 );
[ [ -1, 5 ], [ 0, 1 ] ]
gap> F2 := StarSubdivisionOfIthMaximalCone( F, 1 );
<A fan in \( \mathbb{R}^2 \rangle

gap> IsSmooth( F2 );
true

gap> RayGenerators( F2 );
[ [ -1, 5 ], [ -1, 6 ], [ 0, -1 ], [ 0, 1 ], [ 1, 0 ] ]
```
Chapter 5

Cone

5.1 Cone: Category and Representations

5.1.1 IsCone

\[ \text{IsCone}(M) \] (Category)

Returns: true or false

The GAP category of a cone.

Remember: Every cone is a convex object.

5.2 Cone: Properties

5.2.1 IsRay

\[ \text{IsRay}(\text{cone}) \] (property)

Returns: true or false

Checks if the cone \text{cone} is a ray, i.e. if it has only one ray generator.

5.3 Cone: Attributes

5.3.1 DualCone

\[ \text{DualCone}(\text{cone}) \] (attribute)

Returns: a cone

Returns the dual cone of the cone \text{cone}.

5.3.2 HilbertBasis

\[ \text{HilbertBasis}(\text{cone}) \] (attribute)

Returns: a list

Returns a Hilbert Basis of the cone \text{cone}.
5.3.3 RaysInFacets

- \( \text{RaysInFacets}(\text{cone}) \)
  - **Returns**: a list
  - Returns an incidence matrix for the rays in the facets of the cone \( \text{cone} \). The \( i \)th entry of the result corresponds to the \( i \)th facet, the \( j \)th entry of this is 1 if the \( j \)th ray is in the \( i \)th facet, 0 otherwise.

5.3.4 Facets

- \( \text{Facets}(\text{cone}) \)
  - **Returns**: a list
  - Returns a list of the facets of the cone \( \text{cone} \) as homalg cones.

5.3.5 GridGeneratedByCone

- \( \text{GridGeneratedByCone}(\text{cone}) \)
  - **Returns**: a homalg module
  - Returns the grid generated by the lattice points of the cone \( \text{cone} \) as a homalg module.

5.3.6 FactorGrid

- \( \text{FactorGrid}(\text{cone}) \)
  - **Returns**: a homalg module
  - Returns the factor of the containing grid of the cone \( \text{cone} \) and the grid generated by \( \text{cone} \).

5.3.7 GridGeneratedByOrthogonalCone

- \( \text{GridGeneratedByOrthogonalCone}(\text{cone}) \)
  - **Returns**: a homalg module
  - Returns the grid generated by the lattice points of the orthogonal cone of the cone \( \text{cone} \).

5.3.8 DefiningInequalities

- \( \text{DefiningInequalities}(\text{cone}) \)
  - **Returns**: a list
  - Returns a list of the defining inequalities of the cone \( \text{cone} \).

5.3.9 IsContainedInFan

- \( \text{IsContainedInFan}(\text{cone}) \)
  - **Returns**: a fan
  - If the cone \( \text{cone} \) is constructed as part of a fan, this method returns the fan.

5.3.10 FactorGridMorphism

- \( \text{FactorGridMorphism}(\text{cone}) \)
  - **Returns**: a morphism
  - Returns the morphism to the factor grid of the cone \( \text{cone} \).
5.4 Cone: Methods

5.4.1 IntersectionOfCones

\( \text{IntersectionOfCones(cone1, cone2)} \)

\textbf{Returns:} a cone

If the cones \textit{cone1} and \textit{cone2} share a face, the method returns their intersection.

5.4.2 Contains

\( \text{Contains(cone1, cone2)} \)

\textbf{Returns:} true or false

Returns true if the cone \textit{cone1} contains the cone \textit{cone2}, false otherwise.

5.4.3 StarFan (for a cone)

\( \text{StarFan(cone)} \)

\textbf{Returns:} a fan

Returns the star fan of the cone \textit{cone}, as described in cox, 3.2.7

5.4.4 StarFan (for a cone and a fan)

\( \text{StarFan(cone, fan)} \)

\textbf{Returns:} a fan

Returns the star fan of the fan \textit{fan} along the cone \textit{cone}, as described in cox, 3.2.7

5.4.5 StarSubdivisionOfIthMaximalCone

\( \text{StarSubdivisionOfIthMaximalCone(fan, numb)} \)

\textbf{Returns:} a fan

Returns the star subdivision of the fan \textit{fan} on the \textit{numb}th maximal cone as in cox, 3.3.13.

5.5 Cone: Constructors

5.5.1 Cone (for a ray list)

\( \text{Cone(cone)} \)

\textbf{Returns:} a cone

Returns a cone generated by the rays in \textit{cone}.

5.6 Cone: Examples

5.6.1 Cone example

\begin{verbatim}
gap> C := Cone([[1,2,3],[2,1,1],[1,0,0],[0,1,1]]);  
<A cone in |R^3>  
gap> Length(RayGenerators( C ));  
3  
gap> IsSmooth( C );  
true
\end{verbatim}
true
\texttt{gap> Length( HilbertBasis( C ) );}
3
\texttt{gap> IsSimplicial( C );}
true
\texttt{gap> DC := DualCone( C );}
\texttt{<A cone in |R^3>}
\texttt{gap> Length( HilbertBasis( DC ) );}
3
Chapter 6

Polytope

6.1 Polytope: Category and Representations

6.1.1 IsPolytope

\[ \text{IsPolytope}(M) \]

**Returns:** true or false

The GAP category of a polytope. Every polytope is a convex object. Remember: Every cone is a convex object.

6.2 Polytope: Properties

6.2.1 IsNotEmpty

\[ \text{IsNotEmpty}(poly) \]

**Returns:** true or false

Checks if the polytope \( poly \) is not empty.

6.2.2 IsLatticePolytope

\[ \text{IsLatticePolytope}(poly) \]

**Returns:** true or false

Checks if the polytope \( poly \) is a lattice polytope, i.e. all its vertices are lattice points.

6.2.3 IsVeryAmple

\[ \text{IsVeryAmple}(poly) \]

**Returns:** true or false

Checks if the polytope \( poly \) is very ample.

6.2.4 IsNormalPolytope

\[ \text{IsNormalPolytope}(poly) \]

**Returns:** true or false

Checks if the polytope \( poly \) is normal.
6.2.5 IsSimplicial (for a polytope)

\[ \text{IsSimplicial}(\text{poly}) \]

**(property)**

*Returns:* true or false

Checks if the polytope \textit{poly} is simplicial.

6.2.6 IsSimplePolytope

\[ \text{IsSimplePolytope}(\text{poly}) \]

**(property)**

*Returns:* true or false

Checks if the polytope \textit{poly} is simple.

6.3 Polytope: Attributes

6.3.1 Vertices

\[ \text{Vertices}(\text{poly}) \]

**(attribute)**

*Returns:* a list

Returns the vertices of the polytope \textit{poly}. For reasons, the corresponding tester is HasVerticesOfPolytopes.

6.3.2 LatticePoints

\[ \text{LatticePoints}(\text{poly}) \]

**(attribute)**

*Returns:* a list

Returns the lattice points of the polytope \textit{poly}.

6.3.3 FacetInequalities

\[ \text{FacetInequalities}(\text{poly}) \]

**(attribute)**

*Returns:* a list

Returns the facet inequalities for the polytope \textit{poly}.

6.3.4 VerticesInFacets

\[ \text{VerticesInFacets}(\text{poly}) \]

**(attribute)**

*Returns:* a list

Returns the incidence matrix of vertices and facets of the polytope \textit{poly}.

6.3.5 AffineCone

\[ \text{AffineCone}(\text{poly}) \]

**(attribute)**

*Returns:* a cone

Returns the affine cone of the polytope \textit{poly}.
6.3.6 NormalFan

\( \text{NormalFan(poly)} \)  
\( \text{(attribute)} \)

Returns: a fan

Returns the normal fan of the polytope \( poly \).

6.3.7 RelativeInteriorLatticePoints

\( \text{RelativeInteriorLatticePoints(poly)} \)  
\( \text{(attribute)} \)

Returns: a list

Returns the lattice points in the relative interior of the polytope \( poly \).

6.4 Polytope: Methods

6.4.1 * (for polytopes)

\( \text{*}(polytope1, polytope2) \)  
\( \text{(operation)} \)

Returns: a polytope

Returns the Cartesian product of the polytopes \( polytope1 \) and \( polytope2 \).

6.4.2 #

\( \text{#}(polytope1, polytope2) \)  
\( \text{(operation)} \)

Returns: a polytope

Returns the Minkowski sum of the polytopes \( polytope1 \) and \( polytope2 \).

6.5 Polytope: Constructors

6.5.1 Polytope (for lists of points)

\( \text{Polytope(points)} \)  
\( \text{(operation)} \)

Returns: a polytope

Returns a polytope that is the convex hull of the points \( points \).

6.5.2 PolytopeByInequalities

\( \text{PolytopeByInequalities(ineqs)} \)  
\( \text{(operation)} \)

Returns: a polytope

Returns a polytope defined by the inequalities \( ineqs \).

6.6 Polytope: Examples

6.6.1 Polytope example

Example

\begin{verbatim}
gap> P := Polytope([ [ 2, 0 ], [ 0, 2 ], [ -1, -1 ] ]);  
<A polytope in |R^2>  
gap> IsVeryAmple( P );  
true
\end{verbatim}
Convex

```gap
gap> LatticePoints( P );
[ [ -1, -1 ], [ 0, 0 ], [ 0, 1 ],
  [ 0, 2 ], [ 1, 0 ], [ 1, 1 ], [ 2, 0 ] ]
gap> NFP := NormalFan( P );
<A complete fan in \mathbb{R}^2>
gap> C1 := MaximalCones( NFP )[ 1 ];
<A cone in \mathbb{R}^2>
gap> RayGenerators( C1 );
[ [ -1, -1 ], [ -1, 3 ] ]
gap> IsRegularFan( NFP );
true
```
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