ExamplesForHomalg

Examples for the GAP package homalg

Version 2013.07.05

July 2013

Mohamed Barakat

Simon Görtzen

Markus Lange-Hegermann

(this manual is still under construction)

This manual is best viewed as an HTML document. The latest version is available ONLINE at:

http://homalg.math.rwth-aachen.de/~barakat/ExamplesForHomalg/homalg-project/chap0.html

An OFFLINE version should be included in the documentation subfolder of the package. This package is part of the homalg-project:

http://homalg.math.rwth-aachen.de/index.php/core-packages/examplesforhomalg
Mohamed Barakat  
Email: barakat@mathematik.uni-kl.de  
Homepage: http://www.mathematik.uni-kl.de/~barakat/  
Address: Department of Mathematics,  
University of Kaiserslautern,  
67653 Kaiserslautern,  
Germany  

Simon Görtzen  
Email: simon.goertzen@rwth-aachen.de  
Homepage: http://wwwb.math.rwth-aachen.de/goertzen/  
Address: Lehrstuhl B für Mathematik, RWTH Aachen, Templergraben 64, 52056 Aachen, Germany  

Markus Lange-Hegermann  
Email: markus.lange.hegermann@rwth-aachen.de  
Homepage: http://wwwb.math.rwth-aachen.de/~markus/  
Address: Lehrstuhl B für Mathematik, RWTH Aachen, Templergraben 64, 52056 Aachen, Germany
Copyright

© 2008-2013 by Mohamed Barakat, Simon Goertzen, Markus Lange-Hegermann

This package may be distributed under the terms and conditions of the GNU Public License Version 2.
## Contents

1. Introduction .......................................................... 4
2. Installation of the ExamplesForHomalg Package ............. 5
3. Examples ...................................................................... 6
   3.1 Spectral Filtrations .................................................. 6
   3.2 Commutative Algebra ............................................... 26

References ....................................................................... 28

Index ............................................................................... 29
Chapter 1

Introduction

[Bar10]
Chapter 2

Installation of the ExamplesForHomalg Package

To install this package just extract the package’s archive file to the GAP pkg directory.

By default the ExamplesForHomalg package is not automatically loaded by GAP when it is installed. You must load the package with

LoadPackage("ExamplesForHomalg");

before its functions become available.

Please, send us an e-mail if you have any questions, remarks, suggestions, etc. concerning this package. Also, I would be pleased to hear about applications of this package.

Mohamed Barakat and Simon Görtzen.
Chapter 3

Examples

3.1 Spectral Filtrations

3.1.1 ExtExt

This is Example B.2 in [Bar].

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> wmat := HomalgMatrix( "[
> x*y, y*z, z, 0, 0, 
> x^3*z, x^2*z^2, 0, x*z^2, -z^2, \n> x^4, x^3*z, 0, x^2*z, -x*z, \n> 0, 0, x*y, -y^2, x^2-1,\n> 0, 0, x^2*z, -x*y*z, y*z, \n> 0, 0, x^2*y-x^2, -x*y^2+x*y,y^2-y \n> ]
"");
<A 6 x 5 matrix over an external ring>
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> Y := Hom( Qxyz, W );
<A right module on 5 generators satisfying yet unknown relations>
gap> F := InsertObjectInMultiFunctor( Functor_Hom_for_fp_modules, 2, Y, "TensorY" );
<<The functor TensorY for f.p. modules and their maps over computable rings>
gap> G := LeftDualizingFunctor( Qxyz );
gap> II_E := GrothendieckSpectralSequence( F, G, W );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:
* * * *
* * * *
```
Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[[ -3 .. 0 ], [ 0 .. 3 ]]

Level 0:
* * * *
* * * *
* * * *
* * * *

Level 1:
* * * *
* * * *
* * * *
* * * *

Level 2:
* * s s
* * * *
* * * *
* * * *

Level 3:
* s s s
* s s s
. s *
. . *

Level 4:
s s s s
3.1.2 Purity

This is Example B.3 in [Bar].

\[ \begin{array}{c}
\text{Example} \\
\end{array} \]

\begin{verbatim}
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<An ascending filtration with degrees [-3 .. 0] and graded parts:
  0: <A non-zero left module presented by yet unknown relations for 23 generators>
    -1: <A non-zero left module presented by 37 relations for 22 generators>
    -2: <A non-zero left module presented by 31 relations for 10 generators>
    -3: <A non-zero left module presented by 32 relations for 5 generators>
  of <A non-zero left module presented by 111 relations for 37 generators> gap> ByASmallerPresentation( filt );
<An ascending filtration with degrees [-3 .. 0] and graded parts:
  0: <A non-zero left module presented by 25 relations for 16 generators>
    -1: <A non-zero left module presented by 30 relations for 14 generators>
    -2: <A non-zero left module presented by 18 relations for 7 generators>
    -3: <A non-zero left module presented by 12 relations for 4 generators>
  of <A non-zero left module presented by 48 relations for 20 generators> gap> m := IsomorphismOfFiltration( filt );
<A non-zero isomorphism of left modules>
\end{verbatim}

\begin{verbatim}
3.1.2 Purity

This is Example B.3 in [Bar].

\[ \begin{array}{c}
\text{Example} \\
\end{array} \]

\begin{verbatim}
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
\end{verbatim}

\begin{verbatim}
gap> wmat := HomalgMatrix( "[ \\\n> x*y, y*z, z, 0, 0, \\\n> x^3*z, x^2*z^2, 0, z^2, -z^2, \\\n> x^4, x^3*z, 0, x^2*z, -x*z, \\\n> 0, 0, x*y, -y^2, x^2-1, \\\n> 0, 0, x^2*z, -x*y*z, y*z, \\\n> 0, 0, x^2*y-x^2, -x*y^2+x*y, y^2-y \\\n> ]", 6, 5, Qxyz );
<A 6 x 5 matrix over an external ring>
\end{verbatim}

\begin{verbatim}
gap> W := LeftPresentation( wmat );
<An left module presented by 6 relations for 5 generators>
\end{verbatim}

\begin{verbatim}
gap> filt := PurityFiltration( W );
<The ascending purity filtration with degrees [-3 .. 0] and graded parts:
  0: <A codegree-[ 1, 1] -pure rank 2 left module presented by 3 relations for 4 generators>
    -1: <A codegree-1-pure grade 1 left module presented by 4 relations for 3 generators>
    -2: <A cyclic reflexively pure grade 2 left module presented by 2 relations for a cyclic generator>
\end{verbatim}
-3: \textit{A cyclic reflexively pure grade 3 left module presented by 3 relations for a cyclic generator}

\textit{of a non-pure rank 2 left module presented by 6 relations for 5 generators}:

\texttt{gap> W;}

\texttt{gap> II_E := SpectralSequence( filt );}

\textit{A stable homological spectral sequence with sheets at levels [0..4] each consisting of left modules at bidegrees [-3..0]x [0..3]}

\texttt{gap> Display( II_E );}

The associated transposed spectral sequence:

\begin{verbatim}
Level 0:
  * * * *
  * * * *
  . * * *
  . . * *

Level 1:
  * * * *
  . . . .
  . . . .
  . . . .

Level 2:
  s . . .
  . . . .
  . . . .

Now the spectral sequence of the bicomplex:

\end{verbatim}

\begin{verbatim}
Level 0:
  * * * *
  * * * *
  . * * *
  . . * *

Level 1:
  * * * *

\end{verbatim}
ExamplesForHomalg

----
Level 2:

s . . .
* s . .
. * * .
. . . *

----
Level 3:

s . . .
* s . .
. . s .
. . . *

----
Level 4:

s . . .
. s . .
. . s .
. . . s

\texttt{gap> m := IsomorphismOfFiltration( filt );}
<A non-zero isomorphism of left modules>
\texttt{gap> IsIdenticalObj( Range( m ), W );}
true
\texttt{gap> Source( m );}
<A left module presented by 12 relations for 9 generators (locked)>
\texttt{gap> Display( last );}
0, 0, x, -y,0,1, 0, 0, 0,
x*y,-y*z,-z,0, 0,0, 0, 0, 0,
x^2,-x*z,0, -z,1,0, 0, 0, 0,
0, 0, 0, y,-z,0, 0, 0, 0,
0, 0, 0, 0, x,0, 0, -z, 0, -1,
0, 0, 0, 0, 0,0, -y, -1, 0, 0,
0, 0, 0, 0, 0,-y,x^2-1,0, 0, 0,
0, 0, 0, 0, 0,0, 0, 0, y-1,0,
0, 0, 0, 0, 0,0, 0, 0, z,
0, 0, 0, 0, 0,0, 0, 0, y,
0, 0, 0, 0, 0,0, 0, 0, x

Cokernel of the map
\texttt{Q[x,y,z]^\langle1x12\rangle -> Q[x,y,z]^\langle1x9\rangle},

currently represented by the above matrix
\texttt{gap> Display( filt );}
Degree 0:
0, 0, x, -y,
x*y,-y*z,-z,0,
x^2,-x*z,0, -z

Cokernel of the map

\( \mathbb{Q}[x,y,z]^{(1\times3)} \rightarrow \mathbb{Q}[x,y,z]^{(1\times4)} \),

currently represented by the above matrix

Degree -1:
y,-z,0,
x,0, -z,
0,x, -y,
0,-y,x^2-1

Cokernel of the map

\( \mathbb{Q}[x,y,z]^{(1\times4)} \rightarrow \mathbb{Q}[x,y,z]^{(1\times3)} \),

currently represented by the above matrix

Degree -2:
\( \mathbb{Q}[x,y,z]/< z, y-1 > \)

Degree -3:
\( \mathbb{Q}[x,y,z]/< z, y, x > \)

gap> Display( m );

1, 0, 0, 0, 0,
0, -1, 0, 0, 0,
0, 0, -1, 0, 0,
0, 0, 0, -1, 0,
-x^2,-x*z, 0, -z, 0,
0, 0, x, -y, 0,
0, 0, 0, 0, -1,
0, 0, x^2,-x*y,y,
-x^3,-x^2*z,0, -x*z,z

the map is currently represented by the above 9 x 5 matrix

3.1.3 A3_Purity

This is Example B.4 in [Bar].

gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> A3 := RingOfDerivations( Qxyz, "Dx,Dy,Dz" );
Q[x,y,z]<Dx,Dy,Dz>
gap> nmat := HomalgMatrix( "[ \

0, 0, x, -y,
x*y,-y*z,-z,0,
x^2,-x*z,0, -z

Cokernel of the map

\( \mathbb{Q}[x,y,z]^{(1\times3)} \rightarrow \mathbb{Q}[x,y,z]^{(1\times4)} \),

currently represented by the above matrix

Degree -1:
y,-z,0,
x,0, -z,
0,x, -y,
0,-y,x^2-1

Cokernel of the map

\( \mathbb{Q}[x,y,z]^{(1\times4)} \rightarrow \mathbb{Q}[x,y,z]^{(1\times3)} \),

currently represented by the above matrix

Degree -2:
\( \mathbb{Q}[x,y,z]/< z, y-1 > \)

Degree -3:
\( \mathbb{Q}[x,y,z]/< z, y, x > \)

gap> Display( m );

1, 0, 0, 0, 0,
0, -1, 0, 0, 0,
0, 0, -1, 0, 0,
0, 0, 0, -1, 0,
-x^2,-x*z, 0, -z, 0,
0, 0, x, -y, 0,
0, 0, 0, 0, -1,
0, 0, x^2,-x*y,y,
-x^3,-x^2*z,0, -x*z,z

the map is currently represented by the above 9 x 5 matrix

3.1.3 A3_Purity

This is Example B.4 in [Bar].

gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> A3 := RingOfDerivations( Qxyz, "Dx,Dy,Dz" );
Q[x,y,z]<Dx,Dy,Dz>
gap> nmat := HomalgMatrix( "[ \

\begin{verbatim}
> 3*Dy*Dz-Dz^2+Dx+3*Dy-Dz, 3*Dy*Dz-Dz^2, \\
> Dx*Dz+Dz^2+Dz, 3*Dx*Dy*Dz^2, \\
> -Dx+Dz^2, 0, \\
> 2*Dx*Dz-2*Dx*Dz+Dz^2, Dz^3, \\
> 2*Dx*Dz^2-2*Dx+3*Dx+3*Dz+3*Dz+3*Dz+3*Dz, 2*Dx*Dz^2+3*Dx+3*Dz
\end{verbatim}

\[ A \begin{pmatrix} 8 & 2 \\ 8 & 2 \end{pmatrix} \]

\textit{A left module presented by 8 relations for 2 generators}

\texttt{gap> N := LeftPresentation( nmat );
\textit{A left module presented by 8 relations for 2 generators}}

\texttt{gap> filt := PurityFiltration( N );
\textit{The ascending purity filtration with degrees \([-3 .. 0]\) and graded parts:}}

\texttt{0: \begin{pmatrix} 3*Dy*Dz-Dz^2, 3*Dy*Dz-Dz^2, \\
> Dx*Dz+Dz^2+Dz, 3*Dx*Dy*Dz^2, \\
> -Dx+Dz^2, 0, \\
> 2*Dx*Dz-2*Dx*Dz+Dz^2, Dz^3, \\
> 2*Dx*Dz^2-2*Dx+3*Dx+3*Dz+3*Dz+3*Dz, 2*Dx*Dz^2+3*Dx+3*Dz
\end{pmatrix};
\textit{A zero left module}}

\texttt{-1: \begin{pmatrix} 3*Dy*Dz-Dz^2, 3*Dy*Dz-Dz^2, \\
> Dx*Dz+Dz^2+Dz, 3*Dx*Dy*Dz^2, \\
> -Dx+Dz^2, 0, \\
> 2*Dx*Dz-2*Dx*Dz+Dz^2, Dz^3, \\
> 2*Dx*Dz^2-2*Dx+3*Dx+3*Dz+3*Dz+3*Dz, 2*Dx*Dz^2+3*Dx+3*Dz
\end{pmatrix};
\textit{A cyclic reflexively pure grade 1 left module presented by 1 relation for a cyclic generator}}

\texttt{-2: \begin{pmatrix} 3*Dy*Dz-Dz^2, 3*Dy*Dz-Dz^2, \\
> Dx*Dz+Dz^2+Dz, 3*Dx*Dy*Dz^2, \\
> -Dx+Dz^2, 0, \\
> 2*Dx*Dz-2*Dx*Dz+Dz^2, Dz^3, \\
> 2*Dx*Dz^2-2*Dx+3*Dx+3*Dz+3*Dz+3*Dz, 2*Dx*Dz^2+3*Dx+3*Dz
\end{pmatrix};
\textit{A cyclic reflexively pure grade 2 left module presented by 2 relations for a cyclic generator}}

\texttt{-3: \begin{pmatrix} 3*Dy*Dz-Dz^2, 3*Dy*Dz-Dz^2, \\
> Dx*Dz+Dz^2+Dz, 3*Dx*Dy*Dz^2, \\
> -Dx+Dz^2, 0, \\
> 2*Dx*Dz-2*Dx*Dz+Dz^2, Dz^3, \\
> 2*Dx*Dz^2-2*Dx+3*Dx+3*Dz+3*Dz+3*Dz, 2*Dx*Dz^2+3*Dx+3*Dz
\end{pmatrix};
\textit{A cyclic reflexively pure grade 3 left module presented by 3 relations for a cyclic generator}}

\texttt{gap> II_E := SpectralSequence( filt );
\textit{A stable homological spectral sequence with sheets at levels \([-3 .. 0]\) x \([0 .. 3]\) x \([0 .. 3]\)}

\texttt{gap> Display( II_E );
\textit{The associated transposed spectral sequence:}}

\begin{verbatim}
Level 0:
   * * * *
   . * * *
   . . * *
   . . . *

Level 1:
   * * * *
   . . . .
   . . . .
   . . . .

Level 2:
   * . . .
\end{verbatim}
Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees 
\[
\begin{bmatrix}
[-3 & 0], \\
[0 & 3]
\end{bmatrix}
\]

Level 0:

```
* * * *
.* * *
.* .*
.* .
```

Level 1:

```
* * * *
.* * *
.* .*
.* .
```

Level 2:

```
s . . .
.s . .
.. s .
```

\texttt{gap> m := IsomorphismOfFiltration( filt );}
\texttt{<A non-zero isomorphism of left modules>}
\texttt{gap> IsIdenticalObj( Range( m ), N );}
\texttt{true}
\texttt{gap> Source( m );}
\texttt{<A left module presented by 6 relations for 3 generators (locked)>}
\texttt{gap> Display( last );}
\texttt{Dx,1/3,-1/9*x, 0, Dy, 1/6, 0, Dx, -1/2, 0, 0, Dz, 0, 0, Dy, 0, 0, Dx}

Cokernel of the map
\[ R^{1x6} \rightarrow R^{1x3}, \text{ (for } R := \mathbb{Q}[x,y,z]<Dx,Dy,Dz> \) 

currently represented by the above matrix
\texttt{gap> Display( filt );}
Degree 0:

```
0
```

---------
Degree -1:
\[ \mathbb{Q}[x,y,z]/(D_x, D_y, D_z) / (D_x) \]

Degree -2:
\[ \mathbb{Q}[x,y,z]/(D_y, D_x) \]

Degree -3:
\[ \mathbb{Q}[x,y,z]/(D_z, D_y, D_x) \]

\texttt{gap}>
\begin{verbatim}
Display( m );
1, 1,
3*Dz+3, 3*Dz,
-6*Dz^2+6*Dx-6*Dz,-6*Dz^2
\end{verbatim}
the map is currently represented by the above 3 x 2 matrix

3.1.4 TorExt-Grothendieck

This is Example B.5 in [Bar].

\begin{verbatim}
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> wmat := HomalgMatrix( "[ \n> x*y, y*z, z, 0, 0, 
> x^3*z, x^2*z^2, 0, x*z^2, -z^2, 
> x^4, x^3*z, 0, x^2*z, -x*z, 
> 0, 0, x*y, -y^2, x^2-1, 
> 0, 0, x^2*z, -x*y*z, y*z, 
> 0, 0, x^2*y-x^2, -x*y^2+x*y, y^2-y 
> ]" , 6, 5, Qxyz );
<A 6 x 5 matrix over an external ring>
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> F := InsertObjectInMultiFunctor( Functor_TensorProduct_for_fp_modules, 2, W, "TensorW" );
<br>The functor TensorW for f.p. modules and their maps over computable rings>
gap> G := LeftDualizingFunctor( Qxyz );
<The functor TensorW for f.p. modules and their maps over computable rings>
gap> II_E := GrothendieckSpectralSequence( F, G, W );
<A stable cohomological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]
x [ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a cohomological spectral sequence at bidegrees
[ [ 0 .. 3 ] , [ -3 .. 0 ] ]

Level 0:

* * * *
* * * *
Now the spectral sequence of the bicomplex:

a cohomological spectral sequence at bidegrees

\[ \left[ \left[ -3 .. 0 \right], \left[ 0 .. 3 \right] \right] \]
ExamplesForHomalg

\texttt{filt := FiltrationBySpectralSequence( II_E, 0 );}
\texttt{(A descending filtration with degrees [-3 .. 0] and graded parts:}

-3: \texttt{<A non-zero cyclic torsion left module presented by yet unknown relations for a cyclic generator>}
-2: \texttt{<A non-zero left module presented by 17 relations for 6 generators>}
-1: \texttt{<A non-zero left module presented by 23 relations for 10 generators>}
0: \texttt{<A non-zero left module presented by 13 relations for 10 generators> of}
\texttt{<A left module presented by yet unknown relations for 41 generators>}}
\texttt{gap> ByASmallerPresentation( filt );}
\texttt{(A descending filtration with degrees [-3 .. 0] and graded parts:}

-3: \texttt{<A non-zero cyclic torsion left module presented by 3 relations for a cyclic generator>}
-2: \texttt{<A non-zero left module presented by 12 relations for 4 generators>}
-1: \texttt{<A non-zero left module presented by 18 relations for 8 generators>}
0: \texttt{<A non-zero left module presented by 11 relations for 10 generators> of}
\texttt{<A non-zero left module presented by 21 relations for 12 generators>}}
\texttt{gap> m := IsomorphismOfFiltration( filt );}
\texttt{(A non-zero isomorphism of left modules>}

\texttt{3.1.5 TorExt}

This is Example B.6 in [Bar].

\texttt{gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";}
\texttt{Q[x,y,z]}
\texttt{gap> wmat := HomalgMatrix( "[ \\
> x*y, y*z, z, 0, 0, \\
> x^3*z, x^2*z^2, 0, x*z^2, -z^2, \\
> x^4, x^3*z, 0, x^2*z, -x*z, \\
> 0, 0, x*y, -y^2, x^2-1, \\
> 0, 0, x^2*z, -x*y*z, y*z, \\
> 0, 0, x^2*y-x^2, -x*y^2+x*y, y^2-y \\
> ]", 6, 5, Qxyz );}
\texttt{(A 6 x 5 matrix over an external ring>}
\texttt{gap> W := LeftPresentation( wmat );}
\texttt{(A left module presented by 6 relations for 5 generators>}
\texttt{gap> P := Resolution( W );}
\texttt{(A right acyclic complex containing 3 morphisms of left modules at degrees [ 0 .. 3 ]>}
\texttt{gap> GP := Hom( P );}
\texttt{(A cocomplex containing 3 morphisms of right modules at degrees [ 0 .. 3 ]>}
\texttt{gap> FGP := GP * P;}
\texttt{(A cocomplex containing 3 morphisms of left complexes at degrees [ 0 .. 3 ]>}
\texttt{gap> BC := HomalgBicomplex( FGP );}
\texttt{(A bicocomplex containing left modules at bidegrees [ 0 .. 3 ] x [-3 .. 0 ]>}

\texttt{Example}
\begin{verbatim}
gap> p_degrees := ObjectDegreesOfBicomplex( BC )[1];
[ 0 .. 3 ]
gap> II_E := SecondSpectralSequenceWithFiltration( BC, p_degrees );
< A stable cohomological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ] >
gap> Display( II_E );

The associated transposed spectral sequence:

a cohomological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]

Level 0:
* * * *
* * * *
* * * *
* * * *

Level 1:
* * * *
.....
.....
.....

Level 2:
 s s s s
.....
.....

Now the spectral sequence of the bicomplex:

a cohomological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]

Level 0:
* * * *
* * * *
* * * *

Level 1:
* * * *
* * * *
* * * *

Level 2:
\end{verbatim}
3.1.6 CodegreeOfPurity

This is Example B.7 in \[\text{Bar}\].

```gap
Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
```

```gap
vmat := HomalgMatrix( "[ \\
> 0, 0, x,-z, \\
> x*z,z^2,y,0, \\
> x^2,x*z,0,y \\
> ]" );
```
ExamplesForHomalg

> ]; 3, 4, Qxyz );

\begin{verbatim}
<A 3 x 4 matrix over an external ring>
gap> V := LeftPresentation( vmat );
\end{verbatim}

\begin{verbatim}
<A non-torsion left module presented by 3 relations for 4 generators>
gap> wmat := HomalgMatrix( "[ \\
> 0, 0, x,-y, \ \\
> x*y,y*z,z,0, \ \\
> x^2,x*z,0,z \ \\
> ]", 3, 4, Qxyz );
\end{verbatim}

\begin{verbatim}
<A 3 x 4 matrix over an external ring>
gap> W := LeftPresentation( wmat );
\end{verbatim}

\begin{verbatim}
<A non-torsion left module presented by 3 relations for 4 generators>
\end{verbatim}

\begin{verbatim}
Rank( V );
2
\end{verbatim}

\begin{verbatim}
Rank( W );
2
\end{verbatim}

\begin{verbatim}
ProjectiveDimension( V );
2
\end{verbatim}

\begin{verbatim}
ProjectiveDimension( W );
2
\end{verbatim}

\begin{verbatim}
DegreeOfTorsionFreeness( V );
1
\end{verbatim}

\begin{verbatim}
DegreeOfTorsionFreeness( W );
1
\end{verbatim}

\begin{verbatim}
CodegreeOfPurity( V );
[ 2 ]
\end{verbatim}

\begin{verbatim}
CodegreeOfPurity( W );
[ 1, 1 ]
\end{verbatim}

\begin{verbatim}
filtV := PurityFiltration( V );
\end{verbatim}

\begin{verbatim}
The ascending purity filtration with degrees [-2 .. 0] and graded parts:
0: <A codegree-[ 2 ]-pure rank 2 left module presented by 3 relations for 4 generators>
-1: <A zero left module>
-2: <A zero left module>
of
A codegree-[ 2 ]-pure rank 2 left module presented by 3 relations for 4 generators>
\end{verbatim}

\begin{verbatim}
filtW := PurityFiltration( W );
\end{verbatim}

\begin{verbatim}
The ascending purity filtration with degrees [-2 .. 0] and graded parts:
0: <A codegree-[ 1, 1 ]-pure rank 2 left module presented by 3 relations for 4 generators>
-1: <A zero left module>
-2: <A zero left module>
of
A codegree-[ 1, 1 ]-pure rank 2 left module presented by 3 relations for 4 generators>
\end{verbatim}

\begin{verbatim}
II_EV := SpectralSequence( filtV );
\end{verbatim}

\begin{verbatim}
A stable homological spectral sequence with sheets at levels [ 0 .. 4 ] each consisting of left modules at bidegrees [-3 .. 0]x [ 0 .. 2 ]
\end{verbatim}

\begin{verbatim}
Display( II_EV );
\end{verbatim}
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
\[ [ [ 0 .. 2 ], [ -3 .. 0 ] ] \]
---------
Level 0:
* * *  
* * *  
* * *  
* * *  
---------
Level 1:
* * *  
...  
...  
...  
---------
Level 2:
s ...
...  
...  
...  

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
\[ [ [ -3 .. 0 ], [ 0 .. 2 ] ] \]
---------
Level 0:
* * *  
* * *  
* * *  
---------
Level 1:
* * *  
* * *  
* * *  
---------
Level 2:
* ...  
* ...  
* ...  
---------
Level 3:
* ...  
...  

II_EW := SpectralSequence( filtW );

gap> Display( II_EW );

The associated transposed spectral sequence:

[a homological spectral sequence at bidegrees
[ [ 0 .. 2 ], [ -3 .. 0 ] ]

Level 0:
* * *
* * *
* * *
* * *

Level 1:
* * *
... 
... 
... 

Level 2:
. . .
...
...
...

Now the spectral sequence of the bicomplex:

[a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 2 ] ]

Level 0:
* * * *
* * *
* * *
* * *

Level 1:
* * *
3.1.7 HomHom

This corresponds to the example of Section 2 in [BR06].

```
gap> R := HomalgRingOfIntegersInExternalGAP( ) / 2^8;
Z/( 256 )
gap> Display( R );
<A residue class ring>
gap> M := LeftPresentation( [ 2^5 ], R );
<A cyclic left module presented by an unknown number of relations for a cyclic\generator>
gap> Display( M );
Z/( 256 )/< |
  [ 32 ] |

gap> M;
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> _M := LeftPresentation( [ 2^3 ], R );
<A cyclic left module presented by an unknown number of relations for a cyclic\generator>
gap> Display( _M );
Z/( 256 )/< |
  [ 8 ] |

gap> _M;
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> alpha2 := HomalgMap( [ 1 ], M, _M );
<A "homomorphism" of left modules>
gap> IsMorphism( alpha2 );
true
gap> alpha2;
<A homomorphism of left modules>
gap> Display( alpha2 );
[ [ 1 ] ]
modulo [ 256 ]
```
the map is currently represented by the above 1 x 1 matrix
\[
\begin{bmatrix}
M_\ := \text{Kernel}(\ \text{alpha2} )
\end{bmatrix}
\]
\[
\text{A cyclic left module presented by yet unknown relations for a cyclic generator}
\]
\[
\text{gap} > \text{alpha1} := \text{KernelEmb}(\ \text{alpha2} )
\]
\[
\text{A monomorphism of left modules}
\]
\[
\text{gap} > \text{seq} := \text{HomalgComplex}(\ \text{alpha2} )
\]
\[
\text{An acyclic complex containing a single morphism of left modules at degrees [ 0 .. 1 ]}
\]
\[
\text{gap} > \text{Add}(\ \text{seq},\ \text{alpha1} )
\]
\[
\text{gap} > \text{seq}
\]
\[
\text{A sequence containing 2 morphisms of left modules at degrees [ 0 .. 2 ]}
\]
\[
\text{gap} > \text{IsShortExactSequence}(\ \text{seq} )
\]
\[
\text{true}
\]
\[
\text{gap} > \text{seq}
\]
\[
\text{A short exact sequence containing 2 morphisms of left modules at degrees [ 0 .. 2 ]}
\]
\[
\text{gap} > \text{Display}(\ \text{seq} )
\]
\[
\text{-----------}
\text{at homology degree: 2}
\text{-----------}
\text{[ [ 24 ] ]}
\text{modulo [ 256 ]}
\text{the map is currently represented by the above 1 x 1 matrix}
\text{-----------}
\text{at homology degree: 1}
\text{Z/( 256 )/< [ [ 32 ] ] >}
\text{-----------}
\text{[ [ 1 ] ]}
\text{modulo [ 256 ]}
\text{the map is currently represented by the above 1 x 1 matrix}
\text{-----------}
\text{at homology degree: 0}
\text{Z/( 256 )/< [ [ 8 ] ] >}
\text{-----------}
\text{gap} > \text{K} := \text{LeftPresentation}( \text{[ 2^7 ]}, \text{R} );
\text{A cyclic left module presented by an unknown number of relations for a cyclic generator}
\text{gap} > \text{L} := \text{RightPresentation}( \text{[ 2^4 ]}, \text{R} );
\text{A cyclic right module on a cyclic generator satisfying an unknown number of relations}
\text{gap} > \text{triangle} := \text{LHomHom}( 4, \text{seq}, \text{K}, \text{L}, "t" );
\text{An exact triangle containing 3 morphisms of left complexes at degrees [ 1, 2, 3, 1 ]}
\text{gap} > \text{lehs} := \text{LongSequence}(\ \text{triangle} )
\text{A sequence containing 14 morphisms of left modules at degrees [ 0 .. 14 ]}
\text{gap} > \text{ByASmallerPresentation}(\ \text{lehs} );
A non-zero sequence containing 14 morphisms of left modules at degrees 
[0 .. 14]

\textit{gap} > \texttt{IsExactSequence( lehs );}
false
\textit{gap} > lehs;

A non-zero left acyclic complex containing
14 morphisms of left modules at degrees [0 .. 14]

\textit{gap} > \texttt{Assert( 0, IsLeftAcyclic( lehs ) );}
\textit{gap} > \texttt{Display( lehs );}

at homology degree: 14
\Z/\langle 256 \rangle/< |[4]| >

the map is currently represented by the above 1 x 1 matrix

\textit{---v---}
at homology degree: 13
\Z/\langle 256 \rangle/< |[8]| >

the map is currently represented by the above 1 x 1 matrix

\textit{---v---}
at homology degree: 12
\Z/\langle 256 \rangle/< |[8]| >

the map is currently represented by the above 1 x 1 matrix

\textit{---v---}
at homology degree: 11
\Z/\langle 256 \rangle/< |[4]| >

the map is currently represented by the above 1 x 1 matrix

\textit{---v---}
at homology degree: 10
\Z/\langle 256 \rangle/< |[8]| >

the map is currently represented by the above 1 x 1 matrix

\textit{---v---}
the map is currently represented by the above 1 x 1 matrix
--------------------v-------------------
at homology degree: 9
\( \mathbb{Z}/(256)\langle [8] \rangle \)
--------------------
\( \begin{bmatrix} 2 \end{bmatrix} \)
modulo [256]

the map is currently represented by the above 1 x 1 matrix
--------------------v-------------------
at homology degree: 8
\( \mathbb{Z}/(256)\langle [4] \rangle \)
--------------------
\( \begin{bmatrix} 4 \end{bmatrix} \)
modulo [256]

the map is currently represented by the above 1 x 1 matrix
--------------------v-------------------
at homology degree: 7
\( \mathbb{Z}/(256)\langle [8] \rangle \)
--------------------
\( \begin{bmatrix} 2 \end{bmatrix} \)
modulo [256]

the map is currently represented by the above 1 x 1 matrix
--------------------v-------------------
at homology degree: 6
\( \mathbb{Z}/(256)\langle [8] \rangle \)
--------------------
\( \begin{bmatrix} 2 \end{bmatrix} \)
modulo [256]

the map is currently represented by the above 1 x 1 matrix
--------------------v-------------------
at homology degree: 5
\( \mathbb{Z}/(256)\langle [4] \rangle \)
--------------------
\( \begin{bmatrix} 4 \end{bmatrix} \)
modulo [256]

the map is currently represented by the above 1 x 1 matrix
--------------------v-------------------
at homology degree: 4
\( \mathbb{Z}/(256)\langle [8] \rangle \)
--------------------
\( \begin{bmatrix} 2 \end{bmatrix} \)
modulo [256]
the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 3
\[ \mathbb{Z}/(256)/<[8]> \]
------------------------
\[ \begin{bmatrix} 2 \end{bmatrix} \]
modulo \[ 256 \]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 2
\[ \mathbb{Z}/(256)/<[4]> \]
------------------------
\[ \begin{bmatrix} 8 \end{bmatrix} \]
modulo \[ 256 \]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 1
\[ \mathbb{Z}/(256)/<[16]> \]
------------------------
\[ \begin{bmatrix} 1 \end{bmatrix} \]
modulo \[ 256 \]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 0
\[ \mathbb{Z}/(256)/<[8]> \]
------------------------

3.2 Commutative Algebra

3.2.1 Eliminate

```
Example

R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z,l,m";
Q[x,y,z,l,m]
var := Indeterminates( R );
[ x, y, z, l, m ]
x := var[1];; y := var[2];; z := var[3];; l := var[4];; m := var[5];;
L := [ x*m+l-4, y*m+l-2, z*m-l+1, x^2+y^2+z^2-1, x+y-z ];
[ x*m+1-4, y*m+1-2, z*m-1+1, x^2+y^2+z^2-1, x+y-z ]
e := Eliminate( L, [ l, m ] );
<A ? x 1 matrix over an external ring>
Display( e );
4*y+z,
4*x-5*z,
21*z^2-8
```
gap> I := LeftSubmodule( e );
<A torsion-free (left) ideal given by 3 generators>
gap> Display( I );
4*y+z,  
4*x-5*z,  
21*z^2-8

A (left) ideal generated by the 3 entries of the above matrix

gap> J := LeftSubmodule( "x+y-z, -2*z-3*y+x, x^2+y^2+z^2-1", R );
<A torsion-free (left) ideal given by 3 generators>
gap> I = J;
true
References


Index

ExamplesForHomalg, 4