Endow Commutative Rings with an Abelian Grading

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This manual is best viewed as an HTML document. The latest version is available ONLINE at:

http://homalg.math.rwth-aachen.de/~markus/GradedRingForHomalg/chap0.html

An OFFLINE version should be included in the documentation subfolder of the package.
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Chapter 1

Introduction

1.1 What is the Role of the GradedRingForHomalg Package in the homalg Project?

The homalg project [hpa10] aims at providing a general and abstract framework for homological computations. The package GradedRingForHomalg enables the homalg project to endow commutative rings in homalg with an Abelian grading.

1.2 Functionality

The package GradedRingForHomalg on the one hand builds on the package MatricesForHomalg and on the other hands adds functionality to MatricesForHomalg.

1.3 The Math Behind This Package
Chapter 2

Installation of the GradedRingForHomalg Package

To install this package just extract the package’s archive file to the GAP pkg directory. GradedRingForHomalg also needs the package homalg.

By default the GradedRingForHomalg package is not automatically loaded by GAP when it is installed. You must load the package with

LoadPackage("GradedRingForHomalg");

before its functions become available.

Please, send me e-mail if you have any questions, remarks, suggestions, etc. concerning this package. Also, we would be pleased to hear about applications of this package.

Mohamed Barakat and Markus Lange-Hegermann
Chapter 3

Quick Start
Chapter 4

Graded Rings

The package GradedRingForHomalg defines the classes of graded rings, ring elements and matrices over such rings. These three objects can be used as data structures defined in MatricesForHomalg on which the homalg project can rely to do homological computations over graded rings.

The graded rings most prominently can be used with methods known from general homalg rings. The methods for doing the computations are presented in the appendix (B), since they are not for external use. The new attributes and operations are documented here.

Since the objects implemented here are representations from objects elsewhere in the homalg project (i.e. MatricesForHomalg), we want to stress that there are many other operations in MatricesForHomalg, which can be used in connection with the ones presented here. A few of them can be found in the examples and the appendix of this documentation.

4.1 Graded Rings: Category and Representations

4.1.1 IsHomalgGradedRingRep

\[ \text{IsHomalgGradedRingRep}(R) \] (Representation)

Returns: true or false

The representation of homalg graded rings.

(\text{It is a subrepresentation of the GAP representation IsHomalgRingOrFinitelyPresentedModuleRep.})

Code

```
DeclareRepresentation( "IsHomalgGradedRingRep",
  IsHomalgGradedRing and
  IsHomalgGradedRingOrGradedModuleRep,
  [ "ring" ] );
```

4.1.2 IsHomalgGradedRingElementRep

\[ \text{IsHomalgGradedRingElementRep}(r) \] (Representation)

Returns: true or false

The representation of elements of homalg graded rings.

(\text{It is a representation of the GAP category IsHomalgRingElement.})
4.2 Graded Rings: Constructors

4.2.1 HomalgGradedRingElement (constructor for graded ring elements using numerator and denominator)

\[ \text{HomalgGradedRingElement}(\text{numer}, \text{denom}, R) \]

\[ \text{HomalgGradedRingElement}(\text{numer}, R) \]

Returns: a graded ring element

Creates the graded ring element \( \text{numer} / \text{denom} \) or in the second case \( \text{numer} / 1 \) for the graded ring \( R \). Both \( \text{numer} \) and \( \text{denom} \) may either be a string describing a valid global ring element or from the global ring or computation ring.

4.3 Graded Rings: Attributes

4.3.1 DegreeGroup

\[ \text{DegreeGroup}(S) \]

Returns: a left \( \mathbb{Z} \)-module

The degree Abelian group of the commutative graded ring \( S \).

4.3.2 CommonNonTrivialWeightOfIndeterminates

\[ \text{CommonNonTrivialWeightOfIndeterminates}(S) \]

Returns: a degree

The common nontrivial weight of the indeterminates of the graded ring \( S \) if it exists. Otherwise an error is issued. WARNING: Since the DegreeGroup and WeightsOfIndeterminates are in some cases bound together, you MUST not set the DegreeGroup by hand and let the algorithm create the weights. Set both by hand, set only weights or use the method WeightsOfIndeterminates to set both. Never set the DegreeGroup without the WeightsOfIndeterminates, because it simply wont work!

4.3.3 WeightsOfIndeterminates

\[ \text{WeightsOfIndeterminates}(S) \]

Returns: a list or listlist of integers

The list of degrees of the indeterminates of the graded ring \( S \).

4.3.4 MatrixOfWeightsOfIndeterminates

\[ \text{MatrixOfWeightsOfIndeterminates}(S) \]

Returns: a homalg matrix

A homalg matrix where the list (or listlist) of degrees of the indeterminates of the graded ring \( S \) is stored.
4.4 Graded Rings: Operations and Functions

4.4.1 UnderlyingNonGradedRing (for homalg graded rings)

\[ \text{UnderlyingNonGradedRing}(R) \]

Returns: a homalg ring
Internally there is a ring, in which computations take place.

4.4.2 UnderlyingNonGradedRing (for homalg graded ring elements)

\[ \text{UnderlyingNonGradedRing}(r) \]

Returns: a homalg ring
Internally there is a ring, in which computations take place.

4.4.3 Name (for homalg graded ring elements)

\[ \text{Name}(r) \]

Returns: a string
The name of the graded ring element \( r \).
Chapter 5

Homogeneous Matrices

The package GradedRingForHomalg defines the classes of graded rings, ring elements and homogeneous matrices over such rings. These three objects can be used as data structures defined in MatricesForHomalg on which the homalg project can rely to do homological computations over graded rings.

The graded rings most prominently can be used with methods known from general homalg rings. The methods for doing the computations are presented in the appendix (B), since they are not for external use. The new attributes and operations are documented here.

Since the objects implemented here are representations from objects elsewhere in the homalg project (i.e. MatricesForHomalg), we want to stress that there are many other operations in MatricesForHomalg, which can be used in connection with the ones presented here. A few of them can be found in the examples and the appendix of this documentation.

5.1 Homogeneous Matrices: Category and Representations

5.1.1 IsHomalgMatrixOverGradedRingRep

\[ \text{IsHomalgMatrixOverGradedRingRep}(A) \]

(Representation)

Returns: true or false
The representation of homalg matrices with entries in a homalg graded ring.
(It is a representation of the GAP category IsMatrixOverGradedRing.)

Code

\[
\text{DeclareRepresentation( "IsHomalgMatrixOverGradedRingRep", IsMatrixOverGradedRing, [ ];}
\]

5.2 Homogeneous Matrices: Constructors

5.2.1 MatrixOverGradedRing (constructor for matrices over graded rings)

\[ \text{MatrixOverGradedRing(mat, S)} \]

(function)

Returns: a matrix over a graded ring
Creates a matrix for the graded ring \( S \), where \( \text{mat} \) is a matrix over UnderlyingNonGradedRing(S).
5.3 Homogeneous Matrices: Attributes

5.3.1 DegreesOfEntries

\[\text{DegreesOfEntries}(A)\]  
(ATTRIBUTE)

**Returns:** a listlist of degrees/multi-degrees

The matrix of degrees of the matrix \( A \).

5.3.2 NonTrivialDegreePerRow

\[\text{NonTrivialDegreePerRow}(A[, \text{col\_degrees}])\]  
(ATTRIBUTE)

**Returns:** a list of degrees/multi-degrees

The list of first nontrivial degree per row of the matrix \( A \).

5.3.3 NonTrivialDegreePerColumn

\[\text{NonTrivialDegreePerColumn}(A[, \text{row\_degrees}])\]  
(ATTRIBUTE)

**Returns:** a list of degrees/multi-degrees

The list of first nontrivial degree per column of the matrix \( A \).

5.4 Homogeneous Matrices: Operations and Functions

5.4.1 UnderlyingNonGradedRing (for matrices over graded rings)

\[\text{UnderlyingNonGradedRing}(\text{mat})\]  
(OPERATION)

**Returns:** a homalg ring

The nongraded ring underlying \( \text{HomalgRing}(\text{mat}) \).

5.4.2 SetMatElm (for matrices over graded rings)

\[\text{SetMatElm}(\text{mat}, i, j, r, R)\]  
(OPERATION)

Changes the entry \((i,j)\) of the matrix \( \text{mat} \) to the value \( r \). Here \( R \) is the graded homalg ring involved in these computations.

5.4.3 AddToMatElm (for matrices over graded rings)

\[\text{AddToMatElm}(\text{mat}, i, j, r, R)\]  
(OPERATION)

Changes the entry \((i,j)\) of the matrix \( \text{mat} \) by adding the value \( r \) to it. Here \( R \) is the (graded) homalg ring involved in these computations.

5.4.4 MatElmAsString (for matrices over graded rings)

\[\text{MatElmAsString}(\text{mat}, i, j, R)\]  
(OPERATION)

**Returns:** a string

Returns the entry \((i,j)\) of the matrix \( \text{mat} \) as a string. Here \( R \) is the (graded) homalg ring involved in these computations.
5.4.5 MatElm (for matrices over graded rings)

\[ \text{MatElm}(mat, i, j, R) \]  

\textbf{Returns:} a graded ring element

Returns the entry \((i, j)\) of the matrix \(mat\). Here \(R\) is the (graded) \textit{homalg} ring involved in these computations.
Chapter 6

Examples
Appendix A

The Matrix Tool Operations

The functions listed below are components of the homalgTable object stored in the ring. They are only indirectly accessible through standard methods that invoke them.

A.1 The Tool Operations without a Fallback Method

There are matrix methods for which homalg needs a homalgTable entry for non-internal rings, as it cannot provide a suitable fallback. Below is the list of these homalgTable entries.

A.2 The Tool Operations with a Fallback Method

These are the methods for which it is recommended for performance reasons to have a homalgTable entry for non-internal rings. homalg only provides a generic fallback method.

A.2.1 MonomialMatrix

\[ \text{MonomialMatrix}(d, R) \]

**Returns:** a homalg matrix

The column matrix of \( d \)-th monomials of the homalg graded ring \( R \).

\[
gap> R := \text{HomalgFieldOfRationalsInDefaultCAS( )} * "x,y,z";;
gap> S := \text{GradedRing( R )};;
gap> m := \text{MonomialMatrix( 2, S )};
<\text{A 6 x 1 matrix over a graded ring}>
gap> \text{NrRows( m )};
6
gap> m;
<\text{A 6 x 1 matrix over a graded ring}>
gap> \text{Display( m )};
x^2,
x*y,
x*z,
y^2,
y*z,
z^2
(\text{over a graded ring})\]
A.2.2 RandomMatrixBetweenGradedFreeLeftModules

▷ RandomMatrixBetweenGradedFreeLeftModules(degreesS, degreesT, R) (operation)

Returns: a homalg matrix

A random $r \times c$-matrix between the graded free left modules $R^*(-degreesS) \rightarrow R^*(-degreesT)$, where $r = \text{Length}(degreesS)$ and $c = \text{Length}(degreesT)$.

Example

\begin{verbatim}
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "a,b,c";;
gap> S := GradedRing( R );;
gap> rand := RandomMatrixBetweenGradedFreeLeftModules( [ 2, 3, 4 ], [ 1, 2 ], S );
<A 3 x 2 matrix over a graded ring>
gap> #Display( rand );
gap> #a-2*b+2*c, 2,
gap> #a^2-a*b+b^2-2*b*c+5*c^2, 3*c, 
gap> #2*a^3-3*a^2*b+2*a^2+b^2+a*b+c^2-3*b^2+c^2-3*a^2-4*a*b-3*a*c+c^2
\end{verbatim}

A.2.3 RandomMatrixBetweenGradedFreeRightModules

▷ RandomMatrixBetweenGradedFreeRightModules(degreesT, degreesS, R) (operation)

Returns: a homalg matrix

A random $r \times c$-matrix between the graded free right modules $R^*(-degreesS) \rightarrow R^*(-degreesT)$, where $r = \text{Length}(degreesT)$ and $c = \text{Length}(degreesS)$.

Example

\begin{verbatim}
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "a,b,c" * "x,y,z";;
gap> S := GradedRing( R );;
gap> D := HomalgMatrix( "[ \\
> x,2*y, \\
> y,a-b^2, \\
> z,y-b \\
> ]", 3, 2, S );
<A 3 x 2 matrix over an external ring>
gap> N := HomalgMatrix( "[ \\
> x^2-a*y-z^3, x^3-z^2+y, x*y-b, x*z-c, \\
> x, x*y, a-b, x*a*b \\
> ]", 2, 4, S );
\end{verbatim}

A.2.4 Diff

▷ Diff(D, N) (operation)

Returns: a homalg matrix

If $D$ is a $f \times p$-matrix and $N$ is a $g \times q$-matrix then $H = \text{Diff}(D,N)$ is an $fg \times pq$-matrix whose entry $H[g*(i-1)+j, q*(k-1)+l]$ is the result of differentiating $N[j,l]$ by the differential operator corresponding to $D[i,k]$. (Here we follow the Macaulay2 convention.)

Example

\begin{verbatim}
\end{verbatim}
<A 2 x 4 matrix over an external ring>

```gap
H := Diff( D, N );
</A>

<A 6 x 8 matrix over an external ring>

```gap
Display( H );
```

\[
\begin{align*}
2x, & \quad 3x^2, \quad y, \quad z, \quad -6ay^2, \quad -2z^2, \quad 2x, \quad 0, \\
1, & \quad y, \quad 0, \quad ab, \quad 0, \quad 2x, \quad 0, \quad 0, \\
-3ay^2, & \quad -z^2, \quad x, \quad 0, \quad -y^3, \quad 0, \quad 0, \quad 0, \\
0, & \quad x, \quad 0, \quad 0, \quad 0, \quad 0, \quad 1, \quad bx, \\
0, & \quad -2yz, \quad 0, \quad x, \quad -3ay^2, \quad -z^2, \quad x+1, \quad 0, \\
0, & \quad 0, \quad 0, \quad 0, \quad x, \quad 1, \quad -ax
\end{align*}
\]
Appendix B

Overview of the GradedRingForHomalg Package Source Code

This appendix is included in the documentation to shine some light on the mathematical backgrounds of this Package. Neither is it needed to work with this package nor should the methods presented here be called directly. The functions documented here are entries of the so called ring table and not to be called directly. There are higher level methods in declared and installed in MatricesForHomalg, which call this functions (→ ?MatricesForHomalg:The Basic Matrix Operations).

B.1 The generic Methods

We will present some methods as an example, to show the idea:

B.1.1 BasisOfRowModule (for graded rings)

\[
\text{\tt BasisOfRowModule}(M)
\]

Returns: a distinguished basis (i.e. a distinguished generating set) of the module generated by \( M \)

```
BasisOfRowModule :=
function( M )
    return MatrixOverGradedRing(
        BasisOfRowModule( UnderlyingMatrixOverNonGradedRing( M ) ),
        HomalgRing( M ) );
end,
```

B.1.2 DecideZeroRows (for graded rings)

\[
\text{\tt DecideZeroRows}(A, B)
\]

Returns: a reduced form of \( A \) with respect to \( B \)

```
DecideZeroRows :=
function( A, B )
```

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B.1.3 SyzygiesGeneratorsOfRows (for graded rings)

\[ \text{SyzygiesGeneratorsOfRows}(M) \] (function)

Returns: a distinguished basis of the syzygies of the argument

\[
\text{SyzygiesGeneratorsOfRows} :=
\begin{function}(M)
\text{return MatrixOverGradedRing(}
\text{SyzygiesGeneratorsOfRows( UnderlyingMatrixOverNonGradedRing( M ) ),}
\text{HomalgRing( M ) );}
\end{function}
\]

B.2 Tools

The package \texttt{GradedRingForHomalg} also implements tool functions. These are referred to from \texttt{MatricesForHomalg} automatically. We list the implemented methods here and refer to the \texttt{MatricesForHomalg} documentation (→ ?MatricesForHomalg: The Matrix Tool Operations and ?MatricesForHomalg:RingElement) for details. All tools functions from \texttt{MatricesForHomalg} not listed here are also supported by fallback tools.

- IsZero
- IsOne
- Minus
- DivideByUnit
- IsUnit
- Sum
- Product
- ShallowCopy
- ZeroMatrix
- IdentityMatrix
- AreEqualMatrices
- Involution
- CertainRows
• CertainColumns
• UnionOfRows
• UnionOfColumns
• DiagMat
• KroneckerMat
• MulMat
• AddMat
• SubMat
• Compose
• NrRows
• NrColumns
• IsZeroMatrix
• IsDiagonalMatrix
• ZeroRows
• ZeroColumns
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