MatricesForHomalg

Matrices for the homalg project

Version 2018.01.07

June 2015

Mohamed Barakat

Markus Lange-Hegermann

Vinay Wagh

(*this manual is still under construction*)

This manual is best viewed as an HTML document. The latest version is available ONLINE at:

http://homalg.math.rwth-aachen.de/~barakat/homalg-project/MatricesForHomalg/chap0.h

An OFFLINE version should be included in the documentation subfolder of the package. This package is part of the homalg-project:

http://homalg.math.rwth-aachen.de/index.php/core-packages/matricesforhomalg

Mohamed Barakat
Email: barakat@mathematik.uni-kl.de
Homepage: http://www.mathematik.uni-kl.de/~barakat/
Address: Department of Mathematics,
University of Kaiserslautern,
67653 Kaiserslautern,
Germany

Markus Lange-Hegermann
Email: markus.lange.hegermann@rwth-aachen.de
Homepage: http://wwwb.math.rwth-aachen.de/~markus/
Address: Lehrstuhl B für Mathematik, RWTH Aachen, Templergraben 64, 52056 Aachen, Germany

Vinay Wagh
Email: waghoba@gmail.com
Homepage: http://www.iitg.ernet.in/vinay.wagh/
Address: E-102, Department of Mathematics,
Indian Institute of Technology Guwahati,
Guwahati, Assam, India.
PIN: 781 039.

Copyright

Contents

Chapter 1

Introduction

1.1 What is the role of the MatricesForHomalg package in the homalg project?

1.1.1 MatricesForHomalg provides ...

The package MatricesForHomalg provides:

- rings
- ring elements
- ring maps
- matrices

1.1.2 homalg delegates ...

The package homalg *delegates all* matrix operations as it treats matrices and their rings as *black boxes*. homalg comes with a single predefined class of rings and a single predefined class of matrices over these rings – the so-called internal matrices (→ 5.1.2) over so-called internal rings (→ 3.1.4). An internal matrix (resp. ring) is simply a wrapper containing a GAP-builtin matrix (resp. ring). homalg allows other packages to define further classes or extend existing classes of rings and matrices *together* with their operations. For example:

- The homalg subpackage ResidueClassRingForHomalg (→ Appendix D) defines the classes of residue class rings, residue class ring elements, and matrices over residue class rings. Such a matrix is defined by a matrix over the ambient ring which is nevertheless interpreted modulo the ring relations, i.e. modulo the generators of the defining ideal.
- The package GaussForHomalg extends the class of internal matrices enabling it to wrap sparse matrices provided by the package Gauss. GaussForHomalg delegates the essential part of the matrix creation and all matrix operations to Gauss.
- The package HomalgToCAS defines the classes of so-called external rings and matrices and the package RingsForHomalg delegates the essential part of the matrix creation and all matrix operations to external computer algebra systems like Singular, Macaulay2, Sage, Macaulay2,

MAGMA, Maple, ... The package homalg accesses external matrices via pointers. The pointer of an external matrix is simply its name in the external system. HomalgToCAS chooses these names.

- The package LocalizeRingForHomalg defines the classes of local(ized) rings, local ring elements, and local matrices. A homalg local matrix contains a homalg matrix as a numerator and an element of the global ring as a denominator.

The matrix operations are divided into two classes called “Tools” and “Basic”. The “Tools” operations include addition, subtraction, multiplication, extracting certain rows or columns, stacking, and augmenting matrices (→ Appendix B). The “Basic” operations include the two basic operations in linear algebra needed to solve an inhomogeneous linear system $XA = B$ with coefficients in a not necessarily commutative ring R (→ Appendix A):

- Effectively reducing B modulo A, i.e. effectively deciding if a row (or a set of rows) B lies in the R-span of the rows of the matrix A.

- Computing an R-generating set of row syzygies (=R-relations among the rows) of A, i.e. computing an R-generating set of the left kernel of A. This generating set is then given as the rows of a matrix Y and $YA = 0$.

The first operation is nothing but deciding the solvability of the inhomogeneous system $XA = B$ and if solvable to compute a particular solution X, while the second is to compute an R-generating set for the homogeneous solution space, i.e. the solution space of the homogeneous system $YA = 0$. The above is of course also valid for the column convention.

1.1.3 The black box concept

Now we address the following concerns: Wouldn’t the idea of using algorithms like the Gröbnerbasis algorithm(s) as a black box (→ 1.1.2) contradict the following facts?

- It is known that an efficient Gröbnerbasis algorithm depends on the ring R under consideration. For example the implementation of the algorithm depends on the ground ring (or field) k.

- Often enough highly specialized implementations are used to address specific types of linear systems of equations (occuring in specific homological problems) in order to increase the speed or reduce the space needed for the computations.

The following should clarify the above concerns.

- Since each ring comes with its own black box, the first point is automatically resolved.

- Allow the black box coming with each ring to contain the different available implementations and make them accessible to homalg via standarized names, independent of the computer algebra system used to perform computations.

1.2 This manual

Chapter 2 describes the installation of this package. The remaining chapters are each devoted to one of the MatricesForHomalg objects (→ 1.1.1) with its constructors, properties, attributes, and operations.

Chapter 2

Installation of the MatricesForHomalg Package

To install this package just extract the package's archive file to the GAP `pkg` directory.

By default the MatricesForHomalg package is not automatically loaded by GAP when it is installed. You must load the package with

`LoadPackage`("MatricesForHomalg");

before its functions become available.

Please, send me an e-mail if you have any questions, remarks, suggestions, etc. concerning this package. Also, I would be pleased to hear about applications of this package.

Mohamed Barakat

Chapter 3

Rings

3.1 Rings: Category and Representations

3.1.1 IsHomalgRing

▷ `IsHomalgRing(R)` (Category)

Returns: `true` or `false`

The GAP category of homalg rings.

(It is a subcategory of the GAP categories `IsStructureObject` and `IsHomalgRingOrModule`.)

Code
```
DeclareCategory( "IsHomalgRing",
        IsStructureObject and
        IsRingWithOne and
        IsHomalgRingOrModule );
```

3.1.2 IsPreHomalgRing

▷ `IsPreHomalgRing(R)` (Category)

Returns: `true` or `false`

The GAP category of pre homalg rings.

(It is a subcategory of the GAP category `IsHomalgRing`.)

These are rings with an incomplete `homalgTable`. They provide flexibility for developers to support a wider class of rings, as was necessary for the development of the LocalizeRingForHomalg package. They are not suited for direct usage.

Code
```
DeclareCategory( "IsPreHomalgRing",
        IsHomalgRing );
```

3.1.3 IsHomalgRingElement

▷ `IsHomalgRingElement(r)` (Category)

Returns: `true` or `false`

The GAP category of elements of homalg rings which are not GAP4 built-in.

```
Code
DeclareCategory( "IsHomalgRingElement",
        IsExtAElement and
        IsExtLElement and
        IsExtRElement and
        IsAdditiveElementWithInverse and
        IsMultiplicativeElementWithInverse and
        IsAssociativeElement and
        IsAdditivelyCommutativeElement and
        ## all the above guarantees IsHomalgRingElement => IsRingElement (in GAP4)
        IsAttributeStoringRep );
```

3.1.4 IsHomalgInternalRingRep

▷ `IsHomalgInternalRingRep(R)` (Representation)

Returns: `true` or `false`

The internal representation of homalg rings.

(It is a representation of the GAP category `IsHomalgRing`.)

3.2 Rings: Constructors

This section describes how to construct rings for use with MatricesForHomalg, which exploit the GAP4-built-in abilities to perform the necessary ring operations. By this we also mean necessary matrix operations over such rings. For the purposes of MatricesForHomalg only the ring of integers is properly supported in GAP4. The GAP4 extension packages Gauss and GaussForHomalg extend these built-in abilities to operations with sparse matrices over the ring $\mathbb{Z}/p^n$ for p prime and n positive.

If a ring R is supported in MatricesForHomalg any of its residue class rings R/I is supported as well, provided the ideal I of relations admits a finite set of generators as a left resp. right ideal (→ \/ (3.2.3)). This is immediate for commutative noetherian rings.

3.2.1 HomalgRingOfIntegers (constructor for the integers)

▷ `HomalgRingOfIntegers()` (function)

Returns: a homalg ring

▷ `HomalgRingOfIntegers(c)` (function)

Returns: a homalg ring

The no-argument form returns the ring of integers $\mathbb{Z}$ for homalg.

The one-argument form accepts an integer `c` and returns the ring $\mathbb{Z}/c$ for homalg:

- `c` $= 0$ defaults to $\mathbb{Z}$
- if `c` is a prime power then the package GaussForHomalg is loaded (if it fails to load an error is issued)
- otherwise, the residue class ring constructor / (→ \/ (3.2.3)) is invoked

The operation `SetRingProperties` is automatically invoked to set the ring properties.

If for some reason you don't want to use the GaussForHomalg package (maybe because you didn't install it), then use

HomalgRingOfIntegers( ) / *c*;

but note that the computations will then be considerably slower.

3.2.2 HomalgFieldOfRationals (constructor for the field of rationals)

▷ `HomalgFieldOfRationals()` (function)

Returns: a homalg ring

The package GaussForHomalg is loaded and the field of rationals $\mathbb{Q}$ is returned. If GaussForHomalg fails to load an error is issued.

The operation `SetRingProperties` is automatically invoked to set the ring properties.

3.2.3 \/ (constructor for residue class rings)

▷ `\/(R, ring_rel)` (operation)

Returns: a homalg ring

This is the homalg constructor for residue class rings R/I, where *R* is a homalg ring and $I =$*ring_rel* is the ideal of relations generated by *ring_rel*. *ring_rel* might be:

- a set of ring relations of a left resp. right ideal
- a list of ring elements of *R*
- a ring element of *R*

For noncommutative rings: In the first case the set of ring relations should generate the ideal of relations *I* as left resp. right ideal, and their involutions should generate *I* as right resp. left ideal. If *ring_rel* is not a set of relations, a *left* set of relations is constructed.

The operation `SetRingProperties` is automatically invoked to set the ring properties.

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> Display( ZZ );
<An internal ring>
gap> Z256 := ZZ / 2^8;
Z/( 256 )
gap> Display( Z256 );
<A residue class ring>
gap> Z2 := Z256 / 6;
Z/( 256, 6 )
gap> BasisOfRows( MatrixOfRelations( Z2 ) );
<An unevaluated non-zero 1 x 1 matrix over an internal ring>
gap> Z2;
Z/( 2 )
gap> Display( Z2 );
<A residue class ring>
```

3.3 Rings: Properties

The following properties are declared for homalg rings. Note that (apart from so-called true and immediate methods (→ C.1)) there are no methods installed for ring properties. This means that if the value of the ring property `Prop` is not set for a homalg ring *R*, then

`Prop(` *R* `);`

will cause an error. One can use the usual GAP4 mechanism to check if the value of the property is set or not

`HasProp(` *R* `);`

If you discover that a specific property `Prop` is missing for a certain homalg ring *R* you can it add using the usual GAP4 mechanism

`SetProp(` *R*, true `);`

or

`SetProp(` *R*, false `);`

Be very cautious with setting "missing" properties to homalg objects: If the value you set is mathematically wrong homalg will probably draw wrong conclusions and might return wrong results.

3.3.1 IsZero (for rings)

▷ `IsZero(R)` (property)

Returns: `true` or `false`

Check if the ring *R* is a zero, i.e., if `One(R) =Zero(R)`.

3.3.2 ContainsAField

▷ `ContainsAField(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.3 IsRationalsForHomalg

▷ `IsRationalsForHomalg(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.4 IsFieldForHomalg

▷ `IsFieldForHomalg(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.5 IsDivisionRingForHomalg

▷ `IsDivisionRingForHomalg(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.6 IsIntegersForHomalg

▷ `IsIntegersForHomalg(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.7 IsResidueClassRingOfTheIntegers

▷ `IsResidueClassRingOfTheIntegers(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.8 IsBezoutRing

▷ `IsBezoutRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.9 IsIntegrallyClosedDomain

▷ `IsIntegrallyClosedDomain(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.10 IsUniqueFactorizationDomain

▷ `IsUniqueFactorizationDomain(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.11 IsKaplanskyHermite

▷ `IsKaplanskyHermite(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.12 IsDedekindDomain

▷ `IsDedekindDomain(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.13 IsDiscreteValuationRing

▷ `IsDiscreteValuationRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.14 IsFreePolynomialRing

▷ `IsFreePolynomialRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.15 IsWeylRing

▷ `IsWeylRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.16 IsLocalizedWeylRing

▷ `IsLocalizedWeylRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.17 IsGlobalDimensionFinite

▷ `IsGlobalDimensionFinite(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.18 IsLeftGlobalDimensionFinite

▷ `IsLeftGlobalDimensionFinite(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.19 IsRightGlobalDimensionFinite

▷ `IsRightGlobalDimensionFinite(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.20 HasInvariantBasisProperty

▷ `HasInvariantBasisProperty(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.21 HasLeftInvariantBasisProperty

▷ `HasLeftInvariantBasisProperty(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.22 HasRightInvariantBasisProperty

▷ `HasRightInvariantBasisProperty(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.23 IsLocal

▷ `IsLocal(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.24 IsSemiLocalRing

▷ `IsSemiLocalRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.25 IsIntegralDomain

▷ `IsIntegralDomain(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.26 IsHereditary

▷ `IsHereditary(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.27 IsLeftHereditary

▷ `IsLeftHereditary(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.28 IsRightHereditary

▷ `IsRightHereditary(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.29 IsHermite

▷ `IsHermite(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.30 IsLeftHermite

▷ `IsLeftHermite(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.31 IsRightHermite

▷ `IsRightHermite(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.32 IsNoetherian

▷ `IsNoetherian(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.33 IsLeftNoetherian

▷ `IsLeftNoetherian(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.34 IsRightNoetherian

▷ `IsRightNoetherian(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.35 IsCohenMacaulay

▷ `IsCohenMacaulay(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.36 IsGorenstein

▷ `IsGorenstein(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.37 IsKoszul

▷ `IsKoszul(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.38 IsArtinian (for rings)

▷ `IsArtinian(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.39 IsLeftArtinian

▷ `IsLeftArtinian(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.40 IsRightArtinian

▷ `IsRightArtinian(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.41 IsOreDomain

▷ `IsOreDomain(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.42 IsLeftOreDomain

▷ `IsLeftOreDomain(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.43 IsRightOreDomain

▷ `IsRightOreDomain(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.44 IsPrincipalIdealRing

▷ `IsPrincipalIdealRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.45 IsLeftPrincipalIdealRing

▷ `IsLeftPrincipalIdealRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.46 IsRightPrincipalIdealRing

▷ `IsRightPrincipalIdealRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.47 IsRegular

▷ `IsRegular(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.48 IsFiniteFreePresentationRing

▷ `IsFiniteFreePresentationRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.49 IsLeftFiniteFreePresentationRing

▷ `IsLeftFiniteFreePresentationRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.50 IsRightFiniteFreePresentationRing

▷ `IsRightFiniteFreePresentationRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.51 IsSimpleRing

▷ `IsSimpleRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.52 IsSemiSimpleRing

▷ `IsSemiSimpleRing(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.53 IsSuperCommutative

▷ `IsSuperCommutative(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.54 BasisAlgorithmRespectsPrincipalIdeals

▷ `BasisAlgorithmRespectsPrincipalIdeals(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.55 AreUnitsCentral

▷ `AreUnitsCentral(R)` (property)

Returns: `true` or `false`

R is a ring for homalg.

3.3.56 IsMinusOne

▷ `IsMinusOne(r)` (property)

Returns: `true` or `false`

Check if the ring element *r* is the additive inverse of one.

3.3.57 IsMonic (for homalg ring elements)

▷ `IsMonic(r)` (property)

Returns: `true` or `false`

Check if the homalg ring element *r* is monic.

3.3.58 IsMonicUptoUnit (for homalg ring elements)

▷ `IsMonicUptoUnit(r)` (property)

Returns: `true` or `false`

Check if leading coefficient of the homalg ring element *r* is a unit.

3.3.59 IsLeftRegular (for homalg ring elements)

▷ `IsLeftRegular(r)` (property)

Returns: `true` or `false`

Check if the homalg ring element *r* is left regular.

3.3.60 IsRightRegular (for homalg ring elements)

▷ `IsRightRegular(r)` (property)

Returns: `true` or `false`

Check if the homalg ring element *r* is right regular.

3.3.61 IsRegular (for homalg ring elements)

▷ `IsRegular(r)` (property)

Returns: `true` or `false`

Check if the homalg ring element *r* is regular, i.e. left and right regular.

3.4 Rings: Attributes

3.4.1 Inverse (for homalg ring elements)

▷ `Inverse(r)` (attribute)

Returns: a homalg ring element or fail

The inverse of the homalg ring element *r*.

Example
```
gap> ZZ := HomalgRingOfIntegers( );;
gap> R := ZZ / 2^8;
Z/( 256 )
gap> r := (1/3*One(R)+1/5)+3/7;
|[ 157 ]|
gap> 1 / r;          ## = r^-1;
|[ 181 ]|
gap> s := (1/3*One(R)+2/5)+3/7;
|[ 106 ]|
gap> 1 / s;
fail
```

3.4.2 homalgTable

▷ `homalgTable(R)` (attribute)

Returns: a homalg table

The homalg table of *R* is a ring dictionary, i.e. the translator between homalg and the (specific implementation of the) ring.

Every homalg ring has a homalg table.

3.4.3 RingElementConstructor

▷ `RingElementConstructor(R)` (attribute)

Returns: a function

The constructor of ring elements in the homalg ring *R*.

3.4.4 TypeOfHomalgMatrix

▷ `TypeOfHomalgMatrix(R)` (attribute)

Returns: a type

The GAP4-type of homalg matrices over the homalg ring *R*.

3.4.5 ConstructorForHomalgMatrices

▷ `ConstructorForHomalgMatrices(R)` (attribute)

Returns: a type

The constructor for homalg matrices over the homalg ring *R*.

3.4.6 Zero (for homalg rings)

▷ `Zero(R)` (attribute)

Returns: a homalg ring element

The zero of the homalg ring *R*.

3.4.7 One (for homalg rings)

▷ `One(R)` (attribute)

Returns: a homalg ring element

The one of the homalg ring *R*.

3.4.8 MinusOne

▷ `MinusOne(R)` (attribute)

Returns: a homalg ring element

The minus one of the homalg ring *R*.

3.4.9 ProductOfIndeterminates

▷ `ProductOfIndeterminates(R)` (attribute)

Returns: a homalg ring element

The product of indeterminates of the homalg ring *R*.

3.4.10 RationalParameters

▷ `RationalParameters(R)` (attribute)

Returns: a list of homalg ring elements

The list of rational parameters of the homalg ring *R*.

3.4.11 IndeterminatesOfPolynomialRing

▷ `IndeterminatesOfPolynomialRing(R)` (attribute)

Returns: a list of homalg ring elements

The list of indeterminates of the homalg polynomial ring *R*.

3.4.12 RelativeIndeterminatesOfPolynomialRing

▷ `RelativeIndeterminatesOfPolynomialRing(R)` (attribute)

Returns: a list of homalg ring elements

The list of relative indeterminates of the homalg polynomial ring *R*.

3.4.13 IndeterminateCoordinatesOfRingOfDerivations

▷ `IndeterminateCoordinatesOfRingOfDerivations(R)` (attribute)

Returns: a list of homalg ring elements

The list of indeterminate coordinates of the homalg Weyl ring *R*.

3.4.14 RelativeIndeterminateCoordinatesOfRingOfDerivations

▷ `RelativeIndeterminateCoordinatesOfRingOfDerivations(R)` (attribute)

Returns: a list of homalg ring elements

The list of relative indeterminate coordinates of the homalg Weyl ring R.

3.4.15 IndeterminateDerivationsOfRingOfDerivations

▷ `IndeterminateDerivationsOfRingOfDerivations(R)` (attribute)

Returns: a list of homalg ring elements

The list of indeterminate derivations of the homalg Weyl ring R.

3.4.16 RelativeIndeterminateDerivationsOfRingOfDerivations

▷ `RelativeIndeterminateDerivationsOfRingOfDerivations(R)` (attribute)

Returns: a list of homalg ring elements

The list of relative indeterminate derivations of the homalg Weyl ring R.

3.4.17 IndeterminateAntiCommutingVariablesOfExteriorRing

▷ `IndeterminateAntiCommutingVariablesOfExteriorRing(R)` (attribute)

Returns: a list of homalg ring elements

The list of anti-commuting indeterminates of the homalg exterior ring R.

3.4.18 RelativeIndeterminateAntiCommutingVariablesOfExteriorRing

▷ `RelativeIndeterminateAntiCommutingVariablesOfExteriorRing(R)` (attribute)

Returns: a list of homalg ring elements

The list of anti-commuting relative indeterminates of the homalg exterior ring R.

3.4.19 IndeterminatesOfExteriorRing

▷ `IndeterminatesOfExteriorRing(R)` (attribute)

Returns: a list of homalg ring elements

The list of all indeterminates (commuting and anti-commuting) of the homalg exterior ring R.

3.4.20 CoefficientsRing

▷ `CoefficientsRing(R)` (attribute)

Returns: a homalg ring

The ring of coefficients of the homalg ring R.

3.4.21 KrullDimension

▷ `KrullDimension(R)` (attribute)

Returns: a non-negative integer

The Krull dimension of the commutative homalg ring R.

3.4.22 LeftGlobalDimension

▷ `LeftGlobalDimension(R)` (attribute)

Returns: a non-negative integer

The left global dimension of the homalg ring *R*.

3.4.23 RightGlobalDimension

▷ `RightGlobalDimension(R)` (attribute)

Returns: a non-negative integer

The right global dimension of the homalg ring *R*.

3.4.24 GlobalDimension

▷ `GlobalDimension(R)` (attribute)

Returns: a non-negative integer

The global dimension of the homalg ring *R*. The global dimension is defined, only if the left and right global dimensions coincide.

3.4.25 GeneralLinearRank

▷ `GeneralLinearRank(R)` (attribute)

Returns: a non-negative integer

The general linear rank of the homalg ring *R* ([MR01], 11.1.14).

3.4.26 ElementaryRank

▷ `ElementaryRank(R)` (attribute)

Returns: a non-negative integer

The elementary rank of the homalg ring *R* ([MR01], 11.3.10).

3.4.27 StableRank

▷ `StableRank(R)` (attribute)

Returns: a non-negative integer

The stable rank of the homalg ring *R* ([MR01], 11.3.4).

3.4.28 AssociatedGradedRing

▷ `AssociatedGradedRing(R)` (attribute)

Returns: a homalg ring

The graded ring associated to the filtered ring *R*.

3.5 Rings: Operations and Functions

Chapter 4

Ring Maps

A homalg ring map is a data structure for maps between finitely generated rings. homalg more or less provides the basic declarations and installs the generic methods for ring maps, but it is up to other high level packages to install methods applicable to specific rings. For example, the package Sheaves provides methods for ring maps of (finitely generated) affine rings.

4.1 Ring Maps: Category and Representations

4.1.1 IsHomalgRingMap

▷ `IsHomalgRingMap(`*phi*`)` (Category)

Returns: `true` or `false`

The GAP category of ring maps.

4.1.2 IsHomalgRingSelfMap

▷ `IsHomalgRingSelfMap(`*phi*`)` (Category)

Returns: `true` or `false`

The GAP category of ring self-maps.

(It is a subcategory of the GAP category `IsHomalgRingMap`.)

4.1.3 IsHomalgRingMapRep

▷ `IsHomalgRingMapRep(`*phi*`)` (Representation)

Returns: `true` or `false`

The GAP representation of homalg ring maps.

(It is a representation of the GAP category `IsHomalgRingMap` (4.1.1).)

4.2 Ring Maps: Constructors

4.2.1 RingMap (constructor for ring maps)

▷ `RingMap(`*images, S, T*`)` (operation)

Returns: a homalg ring map

This constructor returns a ring map (homomorphism) of finitely generated rings/algebras. It is represented by the images *images* of the set of generators of the source homalg ring *S* in terms of the generators of the target ring *T* (→ 3.2). Unless the source ring is free *and* given on free ring/algebra generators the returned map will cautiously be indicated using parenthesis: "homomorphism". To verify if the result is indeed a well defined map use `IsMorphism` (4.3.1). If source and target are identical objects, and only then, the ring map is created as a selfmap.

4.3 Ring Maps: Properties

4.3.1 IsMorphism (for ring maps)

▷ `IsMorphism(phi)` (property)

Returns: `true` or `false`

Check if *phi* is a well-defined map, i.e. independent of all involved presentations.

4.3.2 IsIdentityMorphism (for ring maps)

▷ `IsIdentityMorphism(phi)` (property)

Returns: `true` or `false`

Check if the homalg ring map *phi* is the identity morphism.

4.3.3 IsMonomorphism (for ring maps)

▷ `IsMonomorphism(phi)` (property)

Returns: `true` or `false`

Check if the homalg ring map *phi* is a monomorphism.

4.3.4 IsEpimorphism (for ring maps)

▷ `IsEpimorphism(phi)` (property)

Returns: `true` or `false`

Check if the homalg ring map *phi* is an epimorphism.

4.3.5 IsIsomorphism (for ring maps)

▷ `IsIsomorphism(phi)` (property)

Returns: `true` or `false`

Check if the homalg ring map *phi* is an isomorphism.

4.3.6 IsAutomorphism (for ring maps)

▷ `IsAutomorphism(phi)` (property)

Returns: `true` or `false`

Check if the homalg ring map *phi* is an automorphism.

4.4 Ring Maps: Attributes

4.4.1 Source (for ring maps)

▷ `Source(phi)` (attribute)

Returns: a homalg ring

The source of the homalg ring map *phi*.

4.4.2 Range (for ring maps)

▷ `Range(phi)` (attribute)

Returns: a homalg ring

The target (range) of the homalg ring map *phi*.

4.4.3 DegreeOfMorphism (for ring maps)

▷ `DegreeOfMorphism(phi)` (attribute)

Returns: an integer

The degree of the morphism *phi* of graded rings.

(no method installed)

4.4.4 CoordinateRingOfGraph (for ring maps)

▷ `CoordinateRingOfGraph(phi)` (attribute)

Returns: a homalg ring

The coordinate ring of the graph of the ring map *phi*.

4.5 Ring Maps: Operations and Functions

Chapter 5

Matrices

5.1 Matrices: Category and Representations

5.1.1 IsHomalgMatrix

▷ IsHomalgMatrix(*A*) (Category)

Returns: true or false

The GAP category of homalg matrices.

Code
```
DeclareCategory( "IsHomalgMatrix",
        IsMatrixObj and
        IsAttributeStoringRep );
```

5.1.2 IsHomalgInternalMatrixRep

▷ IsHomalgInternalMatrixRep(*A*) (Representation)

Returns: true or false

The internal representation of homalg matrices.

(It is a representation of the GAP category IsHomalgMatrix (5.1.1).)

5.2 Matrices: Constructors

5.2.1 HomalgInitialMatrix (constructor for initial matrices filled with zeros)

▷ HomalgInitialMatrix(*m*, *n*, *R*) (function)

Returns: a homalg matrix

A mutable unevaluated initial $m \times n$ homalg matrix filled with zeros over the homalg ring R. This construction is useful in case one wants to define a matrix by assigning its nonzero entries. The property IsInitialMatrix (5.3.26) is reset as soon as the matrix is evaluated. New computed properties or attributes of the matrix won't be cached, until the matrix is explicitly made immutable using (→ MakeImmutable (**Reference: MakeImmutable**)).

Example
```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> z := HomalgInitialMatrix( 2, 3, ZZ );
<An initial 2 x 3 matrix over an internal ring>
```

```
gap> HasIsZero( z );
false
gap> IsZero( z );
true
gap> z;
<A 2 x 3 mutable matrix over an internal ring>
gap> HasIsZero( z );
false
```

Example

```
gap> n := HomalgInitialMatrix( 2, 3, ZZ );
<An initial 2 x 3 matrix over an internal ring>
gap> SetMatElm( n, 1, 1, "1" );
gap> SetMatElm( n, 2, 3, "1" );
gap> MakeImmutable( n );
<A 2 x 3 matrix over an internal ring>
gap> Display( n );
[ [  1,  0,  0 ],
  [  0,  0,  1 ] ]
gap> IsZero( n );
false
gap> n;
<A non-zero 2 x 3 matrix over an internal ring>
```

5.2.2 HomalgInitialIdentityMatrix (constructor for initial quadratic matrices with ones on the diagonal)

▷ HomalgInitialIdentityMatrix(*m*, *R*) (function)

Returns: a homalg matrix

A mutable unevaluated initial $m \times m$ homalg quadratic matrix with ones on the diagonal over the homalg ring *R*. This construction is useful in case one wants to define an elementary matrix by assigning its off-diagonal nonzero entries. The property IsInitialIdentityMatrix (5.3.27) is reset as soon as the matrix is evaluated. New computed properties or attributes of the matrix won't be cached, until the matrix is explicitly made immutable using (→ MakeImmutable (**Reference: MakeImmutable**)).

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> id := HomalgInitialIdentityMatrix( 3, ZZ );
<An initial identity 3 x 3 matrix over an internal ring>
gap> HasIsOne( id );
false
gap> IsOne( id );
true
gap> id;
<A 3 x 3 mutable matrix over an internal ring>
gap> HasIsOne( id );
false
```

Example

```
gap> e := HomalgInitialIdentityMatrix( 3, ZZ );
<An initial identity 3 x 3 matrix over an internal ring>
```

```
gap> SetMatElm( e, 1, 2, "1" );
gap> SetMatElm( e, 2, 1, "-1" );
gap> MakeImmutable( e );
<A 3 x 3 matrix over an internal ring>
gap> Display( e );
[ [   1,   1,   0 ],
  [  -1,   1,   0 ],
  [   0,   0,   1 ] ]
gap> IsOne( e );
false
gap> e;
<A 3 x 3 matrix over an internal ring>
```

5.2.3 HomalgZeroMatrix (constructor for zero matrices)

▷ HomalgZeroMatrix(*m*, *n*, *R*) (function)

Returns: a homalg matrix

An immutable unevaluated $m \times n$ homalg zero matrix over the homalg ring R.

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> z := HomalgZeroMatrix( 2, 3, ZZ );
<An unevaluated 2 x 3 zero matrix over an internal ring>
gap> Display( z );
[ [  0,  0,  0 ],
  [  0,  0,  0 ] ]
gap> z;
<A 2 x 3 zero matrix over an internal ring>
```

5.2.4 HomalgIdentityMatrix (constructor for identity matrices)

▷ HomalgIdentityMatrix(*m*, *R*) (function)

Returns: a homalg matrix

An immutable unevaluated $m \times m$ homalg identity matrix over the homalg ring R.

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> id := HomalgIdentityMatrix( 3, ZZ );
<An unevaluated 3 x 3 identity matrix over an internal ring>
gap> Display( id );
[ [  1,  0,  0 ],
  [  0,  1,  0 ],
  [  0,  0,  1 ] ]
gap> id;
<A 3 x 3 identity matrix over an internal ring>
```

5.2.5 HomalgVoidMatrix (constructor for void matrices)

▷ HomalgVoidMatrix([*m*,][*n*,]*R*) (function)

Returns: a homalg matrix

A void $m \times n$ homalg matrix.

5.2.6 HomalgMatrix (constructor for matrices using a listlist)

▷ HomalgMatrix(*llist*, *R*) (function)
▷ HomalgMatrix(*list*, *m*, *n*, *R*) (function)
▷ HomalgMatrix(*str_llist*, *R*) (function)
▷ HomalgMatrix(*str_list*, *m*, *n*, *R*) (function)

Returns: a homalg matrix

An immutable evaluated $m \times n$ homalg matrix over the homalg ring R.

Example
```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> m := HomalgMatrix( [ [ 1, 2, 3 ], [ 4, 5, 6 ] ], ZZ );
<A 2 x 3 matrix over an internal ring>
gap> Display( m );
[ [  1,  2,  3 ],
  [  4,  5,  6 ] ]
```

Example
```
gap> m := HomalgMatrix( [ [ 1, 2, 3 ], [ 4, 5, 6 ] ], 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
gap> Display( m );
[ [  1,  2,  3 ],
  [  4,  5,  6 ] ]
```

Example
```
gap> m := HomalgMatrix( [ 1, 2, 3,   4, 5, 6 ], 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
gap> Display( m );
[ [  1,  2,  3 ],
  [  4,  5,  6 ] ]
```

Example
```
gap> m := HomalgMatrix( "[ [ 1, 2, 3 ], [ 4, 5, 6 ] ]", ZZ );
<A 2 x 3 matrix over an internal ring>
gap> Display( m );
[ [  1,  2,  3 ],
  [  4,  5,  6 ] ]
```

Example
```
gap> m := HomalgMatrix( "[ [ 1, 2, 3 ], [ 4, 5, 6 ] ]", 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
gap> Display( m );
[ [  1,  2,  3 ],
  [  4,  5,  6 ] ]
```

It is nevertheless recommended to use the following form to create homalg matrices. This form can also be used to define external matrices. Since whitespaces (→ **Reference: Whitespaces**) are ignored, they can be used as optical delimiters:

Example
```
gap> m := HomalgMatrix( "[ 1, 2, 3,   4, 5, 6 ]", 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
gap> Display( m );
[ [  1,  2,  3 ],
  [  4,  5,  6 ] ]
```

One can split the input string over several lines using the backslash character '\' to end each line

Example
```
gap> m := HomalgMatrix( "[ \
> 1, 2, 3, \
> 4, 5, 6  \
> ]", 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
gap> Display( m );
[ [  1,  2,  3 ],
  [  4,  5,  6 ] ]
```

5.2.7 HomalgDiagonalMatrix (constructor for diagonal matrices)

▷ HomalgDiagonalMatrix(*diag*, *R*) (function)

Returns: a homalg matrix

An immutable unevaluated diagonal homalg matrix over the homalg ring *R*. The diagonal consists of the entries of the list *diag*.

Example
```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> d := HomalgDiagonalMatrix( [ 1, 2, 3 ], ZZ );
<An unevaluated diagonal 3 x 3 matrix over an internal ring>
gap> Display( d );
[ [  1,  0,  0 ],
  [  0,  2,  0 ],
  [  0,  0,  3 ] ]
gap> d;
<A diagonal 3 x 3 matrix over an internal ring>
```

5.2.8 * (copy a matrix over a different ring)

▷ *(*R*, *mat*) (operation)

▷ *(*mat*, *R*) (operation)

Returns: a homalg matrix

An immutable evaluated homalg matrix over the homalg ring *R* having the same entries as the matrix *mat*. Syntax: *R* * *mat* or *mat* * *R*

Example
```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> Z4 := ZZ / 4;
Z/( 4 )
gap> Display( Z4 );
<A residue class ring>
gap> d := HomalgDiagonalMatrix( [ 2 .. 4 ], ZZ );
<An unevaluated diagonal 3 x 3 matrix over an internal ring>
gap> d2 := Z4 * d; ## or d2 := d * Z4;
<A 3 x 3 matrix over a residue class ring>
gap> Display( d2 );
[ [  2,  0,  0 ],
  [  0,  3,  0 ],
  [  0,  0,  4 ] ]
```

```
modulo [ 4 ]
gap> d;
<A diagonal 3 x 3 matrix over an internal ring>
gap> ZeroRows( d );
[  ]
gap> ZeroRows( d2 );
[ 3 ]
gap> d;
<A non-zero diagonal 3 x 3 matrix over an internal ring>
gap> d2;
<A non-zero 3 x 3 matrix over a residue class ring>
```

5.3 Matrices: Properties

5.3.1 IsZero (for matrices)

▷ `IsZero(A)` (property)

Returns: `true` or `false`

Check if the homalg matrix *A* is a zero matrix, taking possible ring relations into account. (for the installed standard method see `IsZeroMatrix` (B.1.16))

Example
```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> A := HomalgMatrix( "[ 2 ]", ZZ );
<A 1 x 1 matrix over an internal ring>
gap> Z2 := ZZ / 2;
Z/( 2 )
gap> A := Z2 * A;
<A 1 x 1 matrix over a residue class ring>
gap> Display( A );
[ [  2 ] ]

modulo [ 2 ]
gap> IsZero( A );
true
```

5.3.2 IsOne

▷ `IsOne(A)` (property)

Returns: `true` or `false`

Check if the homalg matrix *A* is an identity matrix, taking possible ring relations into account. (for the installed standard method see `IsIdentityMatrix` (B.2.2))

5.3.3 IsUnitFree

▷ `IsUnitFree(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.4 IsPermutationMatrix

▷ `IsPermutationMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.5 IsSpecialSubidentityMatrix

▷ `IsSpecialSubidentityMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.6 IsSubidentityMatrix

▷ `IsSubidentityMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.7 IsLeftRegular

▷ `IsLeftRegular(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.8 IsRightRegular

▷ `IsRightRegular(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.9 IsInvertibleMatrix

▷ `IsInvertibleMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.10 IsLeftInvertibleMatrix

▷ `IsLeftInvertibleMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.11 IsRightInvertibleMatrix

▷ `IsRightInvertibleMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.12 IsEmptyMatrix

▷ `IsEmptyMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.13 IsDiagonalMatrix

▷ `IsDiagonalMatrix(A)` (property)

Returns: `true` or `false`

Check if the homalg matrix *A* is an identity matrix, taking possible ring relations into account.
(for the installed standard method see `IsDiagonalMatrix` (B.2.3))

5.3.14 IsScalarlMatrix

▷ `IsScalarlMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.15 IsUpperTriangularMatrix

▷ `IsUpperTriangularMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.16 IsLowerTriangularMatrix

▷ `IsLowerTriangularMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.17 IsStrictUpperTriangularMatrix

▷ `IsStrictUpperTriangularMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.18 IsStrictLowerTriangularMatrix

▷ `IsStrictLowerTriangularMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.19 IsUpperStairCaseMatrix

▷ `IsUpperStairCaseMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.20 IsLowerStairCaseMatrix

▷ `IsLowerStairCaseMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.21 IsTriangularMatrix

▷ `IsTriangularMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.22 IsBasisOfRowsMatrix

▷ `IsBasisOfRowsMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.23 IsBasisOfColumnsMatrix

▷ `IsBasisOfColumnsMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.24 IsReducedBasisOfRowsMatrix

▷ `IsReducedBasisOfRowsMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.25 IsReducedBasisOfColumnsMatrix

▷ `IsReducedBasisOfColumnsMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.26 IsInitialMatrix

▷ `IsInitialMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.27 IsInitialIdentityMatrix

▷ `IsInitialIdentityMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.3.28 IsVoidMatrix

▷ `IsVoidMatrix(A)` (property)

Returns: `true` or `false`

A is a homalg matrix.

5.4 Matrices: Attributes

5.4.1 NrRows

▷ `NrRows(A)` (attribute)

Returns: a nonnegative integer

The number of rows of the matrix *A*.

(for the installed standard method see `NrRows` (B.1.17))

5.4.2 NrColumns

▷ `NrColumns(A)` (attribute)

Returns: a nonnegative integer

The number of columns of the matrix *A*.

(for the installed standard method see `NrColumns` (B.1.18))

5.4.3 DeterminantMat

▷ `DeterminantMat(A)` (attribute)

Returns: a ring element

The determinant of the quadratic matrix *A*.

You can invoke it with `Determinant( A )`.

(for the installed standard method see `Determinant` (B.1.19))

5.4.4 ZeroRows

▷ `ZeroRows(A)` (attribute)

Returns: a (possibly empty) list of positive integers

The list of zero rows of the matrix *A*.

(for the installed standard method see `ZeroRows` (B.2.4))

5.4.5 ZeroColumns

▷ `ZeroColumns(A)` (attribute)

Returns: a (possibly empty) list of positive integers

The list of zero columns of the matrix *A*.

(for the installed standard method see `ZeroColumns` (B.2.5))

5.4.6 NonZeroRows

▷ `NonZeroRows(A)` (attribute)

Returns: a (possibly empty) list of positive integers

The list of nonzero rows of the matrix *A*.

5.4.7 NonZeroColumns

▷ `NonZeroColumns(A)` (attribute)

Returns: a (possibly empty) list of positive integers

The list of nonzero columns of the matrix `A`.

5.4.8 PositionOfFirstNonZeroEntryPerRow

▷ `PositionOfFirstNonZeroEntryPerRow(A)` (attribute)

Returns: a list of nonnegative integers

The list of positions of the first nonzero entry per row of the matrix `A`, else zero.

5.4.9 PositionOfFirstNonZeroEntryPerColumn

▷ `PositionOfFirstNonZeroEntryPerColumn(A)` (attribute)

Returns: a list of nonnegative integers

The list of positions of the first nonzero entry per column of the matrix `A`, else zero.

5.4.10 RowRankOfMatrix

▷ `RowRankOfMatrix(A)` (attribute)

Returns: a nonnegative integer

The row rank of the matrix `A`.

5.4.11 ColumnRankOfMatrix

▷ `ColumnRankOfMatrix(A)` (attribute)

Returns: a nonnegative integer

The column rank of the matrix `A`.

5.4.12 LeftInverse

▷ `LeftInverse(M)` (attribute)

Returns: a homalg matrix

A left inverse C of the matrix `M`. If no left inverse exists then `false` is returned. (→ `RightDivide` (5.5.45))

(for the installed standard method see `LeftInverse` (5.5.2))

5.4.13 RightInverse

▷ `RightInverse(M)` (attribute)

Returns: a homalg matrix

A right inverse C of the matrix `M`. If no right inverse exists then `false` is returned. (→ `LeftDivide` (5.5.46))

(for the installed standard method see `RightInverse` (5.5.3))

5.4.14 CoefficientsOfUnreducedNumeratorOfHilbertPoincareSeries

▷ `CoefficientsOfUnreducedNumeratorOfHilbertPoincareSeries(A)` (attribute)

Returns: a list of integers

A is a homalg matrix (row convention).

5.4.15 CoefficientsOfNumeratorOfHilbertPoincareSeries

▷ `CoefficientsOfNumeratorOfHilbertPoincareSeries(A)` (attribute)

Returns: a list of integers

A is a homalg matrix (row convention).

5.4.16 UnreducedNumeratorOfHilbertPoincareSeries

▷ `UnreducedNumeratorOfHilbertPoincareSeries(A)` (attribute)

Returns: a univariate polynomial with rational coefficients

A is a homalg matrix (row convention).

5.4.17 NumeratorOfHilbertPoincareSeries

▷ `NumeratorOfHilbertPoincareSeries(A)` (attribute)

Returns: a univariate polynomial with rational coefficients

A is a homalg matrix (row convention).

5.4.18 HilbertPoincareSeries

▷ `HilbertPoincareSeries(A)` (attribute)

Returns: a univariate rational function with rational coefficients

A is a homalg matrix (row convention).

5.4.19 HilbertPolynomial

▷ `HilbertPolynomial(A)` (attribute)

Returns: a univariate polynomial with rational coefficients

A is a homalg matrix (row convention).

5.4.20 AffineDimension

▷ `AffineDimension(A)` (attribute)

Returns: a nonnegative integer

A is a homalg matrix (row convention).

5.4.21 AffineDegree

▷ `AffineDegree(A)` (attribute)

Returns: a nonnegative integer

A is a homalg matrix (row convention).

5.4.22 ProjectiveDegree

▷ `ProjectiveDegree(A)` (attribute)

Returns: a nonnegative integer

A is a homalg matrix (row convention).

5.4.23 ConstantTermOfHilbertPolynomialn

▷ `ConstantTermOfHilbertPolynomialn(A)` (attribute)

Returns: an integer

A is a homalg matrix (row convention).

5.4.24 MatrixOfSymbols

▷ `MatrixOfSymbols(A)` (attribute)

Returns: an integer

A is a homalg matrix.

5.5 Matrices: Operations and Functions

5.5.1 HomalgRing (for matrices)

▷ `HomalgRing(mat)` (operation)

Returns: a homalg ring

The homalg ring of the homalg matrix *mat*.

Example
```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> d := HomalgDiagonalMatrix( [ 2 .. 4 ], ZZ );
<An unevaluated diagonal 3 x 3 matrix over an internal ring>
gap> R := HomalgRing( d );
Z
gap> IsIdenticalObj( R, ZZ );
true
```

5.5.2 LeftInverse (for matrices)

▷ `LeftInverse(RI)` (method)

Returns: a homalg matrix or fail

The left inverse of the matrix *RI*. The lazy version of this operation is `LeftInverseLazy` (5.5.4). (→ `RightDivide` (5.5.45))

Code
```
InstallMethod( LeftInverse,
        "for homalg matrices",
        [ IsHomalgMatrix ],

  function( RI )
    local Id, LI;

    Id := HomalgIdentityMatrix( NrColumns( RI ), HomalgRing( RI ) );
```

```
    LI := RightDivide( Id, RI );        ## ( cf. [BR08, Subsection 3.1.3] )

    ## CAUTION: for the following SetXXX RightDivide is assumed
    ## NOT to be lazy evaluated!!!

    SetIsLeftInvertibleMatrix( RI, IsHomalgMatrix( LI ) );

    if IsBool( LI ) then
        return fail;
    fi;

    if HasIsInvertibleMatrix( RI ) and IsInvertibleMatrix( RI ) then
        SetIsInvertibleMatrix( LI, true );
    else
        SetIsRightInvertibleMatrix( LI, true );
    fi;

    SetRightInverse( LI, RI );

    SetNrColumns( LI, NrRows( RI ) );

    if NrRows( RI ) = NrColumns( RI ) then
        ## a left inverse of a ring element is unique
        ## and coincides with the right inverse
        SetRightInverse( RI, LI );
        SetLeftInverse( LI, RI );
    fi;

    return LI;

end );
```

5.5.3 RightInverse (for matrices)

▷ RightInverse(*LI*) (method)

Returns: a homalg matrix or fail

The right inverse of the matrix *LI*. The lazy version of this operation is RightInverseLazy (5.5.5). (→ LeftDivide (5.5.46))

Code

```
InstallMethod( RightInverse,
        "for homalg matrices",
        [ IsHomalgMatrix ],

  function( LI )
    local Id, RI;

    Id := HomalgIdentityMatrix( NrRows( LI ), HomalgRing( LI ) );

    RI := LeftDivide( LI, Id );        ## ( cf. [BR08, Subsection 3.1.3] )

    ## CAUTION: for the following SetXXX LeftDivide is assumed
```

```
    ## NOT to be lazy evaluated!!!

    SetIsRightInvertibleMatrix( LI, IsHomalgMatrix( RI ) );

    if IsBool( RI ) then
        return fail;
    fi;

    if HasIsInvertibleMatrix( LI ) and IsInvertibleMatrix( LI ) then
        SetIsInvertibleMatrix( RI, true );
    else
        SetIsLeftInvertibleMatrix( RI, true );
    fi;

    SetLeftInverse( RI, LI );

    SetNrRows( RI, NrColumns( LI ) );

    if NrRows( LI ) = NrColumns( LI ) then
        ## a right inverse of a ring element is unique
        ## and coincides with the left inverse
        SetLeftInverse( LI, RI );
        SetRightInverse( RI, LI );
    fi;

    return RI;

end );
```

5.5.4 LeftInverseLazy (for matrices)

▷ `LeftInverseLazy(M)` (operation)

Returns: a homalg matrix

A lazy evaluated left inverse *C* of the matrix `M`. If no left inverse exists then `Eval( C )` will issue an error.

(for the installed standard method see `Eval` (C.4.5))

5.5.5 RightInverseLazy (for matrices)

▷ `RightInverseLazy(M)` (operation)

Returns: a homalg matrix

A lazy evaluated right inverse *C* of the matrix `M`. If no right inverse exists then `Eval( C )` will issue an error.

(for the installed standard method see `Eval` (C.4.6))

5.5.6 Involution (for matrices)

▷ `Involution(M)` (method)

Returns: a homalg matrix

The twisted transpose of the homalg matrix `M`.

(for the installed standard method see `Eval` (C.4.7))

5.5.7 CertainRows (for matrices)

▷ `CertainRows(M, plist)` (method)

Returns: a homalg matrix

The matrix of which the i-th row is the k-th row of the homalg matrix `M`, where $k =$`plist`$[i]$.

(for the installed standard method see `Eval` (C.4.8))

5.5.8 CertainColumns (for matrices)

▷ `CertainColumns(M, plist)` (method)

Returns: a homalg matrix

The matrix of which the j-th column is the l-th column of the homalg matrix `M`, where $l =$`plist`$[i]$.

(for the installed standard method see `Eval` (C.4.9))

5.5.9 UnionOfRows (for matrices)

▷ `UnionOfRows(A, B)` (method)

Returns: a homalg matrix

Stack the two homalg matrices `A` and `B`.

(for the installed standard method see `Eval` (C.4.10))

5.5.10 UnionOfColumns (for matrices)

▷ `UnionOfColumns(A, B)` (method)

Returns: a homalg matrix

Augment the two homalg matrices `A` and `B`.

(for the installed standard method see `Eval` (C.4.11))

5.5.11 DiagMat (for matrices)

▷ `DiagMat(list)` (method)

Returns: a homalg matrix

Build the block diagonal matrix out of the homalg matrices listed in `list`. An error is issued if `list` is empty or if one of the arguments is not a homalg matrix.

(for the installed standard method see `Eval` (C.4.12))

5.5.12 KroneckerMat (for matrices)

▷ `KroneckerMat(A, B)` (method)

Returns: a homalg matrix

The Kronecker (or tensor) product of the two homalg matrices `A` and `B`.

(for the installed standard method see `Eval` (C.4.13))

5.5.13 * (for ring elements and matrices)

▷ *(a, A) (method)

Returns: a homalg matrix

The product of the ring element a with the homalg matrix A (enter: a * A;).

(for the installed standard method see Eval (C.4.14))

5.5.14 \+ (for matrices)

▷ \+(A, B) (method)

Returns: a homalg matrix

The sum of the two homalg matrices A and B (enter: A + B;).

(for the installed standard method see Eval (C.4.15))

5.5.15 \- (for matrices)

▷ \-(A, B) (method)

Returns: a homalg matrix

The difference of the two homalg matrices A and B (enter: A - B;).

(for the installed standard method see Eval (C.4.16))

5.5.16 * (for composable matrices)

▷ *(A, B) (method)

Returns: a homalg matrix

The matrix product of the two homalg matrices A and B (enter: A * B;).

(for the installed standard method see Eval (C.4.17))

5.5.17 \= (for matrices)

▷ \=(A, B) (operation)

Returns: true or false

Check if the homalg matrices A and B are equal (enter: A = B;), taking possible ring relations into account.

(for the installed standard method see AreEqualMatrices (B.2.1))

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> A := HomalgMatrix( "[ 1 ]", ZZ );
<A 1 x 1 matrix over an internal ring>
gap> B := HomalgMatrix( "[ 3 ]", ZZ );
<A 1 x 1 matrix over an internal ring>
gap> Z2 := ZZ / 2;
Z/( 2 )
gap> A := Z2 * A;
<A 1 x 1 matrix over a residue class ring>
gap> B := Z2 * B;
<A 1 x 1 matrix over a residue class ring>
gap> Display( A );
[ [  1 ] ]
```

```
modulo [ 2 ]
gap> Display( B );
[ [  3 ] ]

modulo [ 2 ]
gap> A = B;
true
```

5.5.18 GetColumnIndependentUnitPositions (for matrices)

▷ `GetColumnIndependentUnitPositions(A, poslist)` (operation)

Returns: a (possibly empty) list of pairs of positive integers

The list of column independet unit position of the matrix *A*. We say that a unit $A[i,k]$ is column independet from the unit $A[l,j]$ if $i > l$ and $A[l,k] = 0$. The rows are scanned from top to bottom and within each row the columns are scanned from right to left searching for new units, column independent from the preceding ones. If $A[i,k]$ is a new column independent unit then $[i,k]$ is added to the output list. If *A* has no units the empty list is returned.

(for the installed standard method see `GetColumnIndependentUnitPositions` (B.2.6))

5.5.19 GetRowIndependentUnitPositions (for matrices)

▷ `GetRowIndependentUnitPositions(A, poslist)` (operation)

Returns: a (possibly empty) list of pairs of positive integers

The list of row independet unit position of the matrix *A*. We say that a unit $A[k,j]$ is row independet from the unit $A[i,l]$ if $j > l$ and $A[k,l] = 0$. The columns are scanned from left to right and within each column the rows are scanned from bottom to top searching for new units, row independent from the preceding ones. If $A[k,j]$ is a new row independent unit then $[j,k]$ (yes $[j,k]$) is added to the output list. If *A* has no units the empty list is returned.

(for the installed standard method see `GetRowIndependentUnitPositions` (B.2.7))

5.5.20 GetUnitPosition (for matrices)

▷ `GetUnitPosition(A, poslist)` (operation)

Returns: a (possibly empty) list of pairs of positive integers

The position $[i,j]$ of the first unit $A[i,j]$ in the matrix *A*, where the rows are scanned from top to bottom and within each row the columns are scanned from left to right. If $A[i,j]$ is the first occurrence of a unit then the position pair $[i,j]$ is returned. Otherwise `fail` is returned.

(for the installed standard method see `GetUnitPosition` (B.2.8))

5.5.21 Eliminate

▷ `Eliminate(rel, indets)` (operation)

Returns: a homalg matrix

Eliminate the independents *indets* from the matrix (or list of ring elements) *rel*, i.e. compute a generating set of the ideal defined as the intersection of the ideal generated by the entries of the list *rel* with the subring generated by all indeterminates except those in *indets*. by the list of indeterminates *indets*.

5.5.22 BasisOfRowModule (for matrices)

▷ `BasisOfRowModule(M)` (operation)

Returns: a homalg matrix

Let R be the ring over which `M` is defined (R :=`HomalgRing( M )`) and S be the row span of `M`, i.e. the R-submodule of the free module $R^{(1\times NrColumns(M))}$ spanned by the rows of `M`. A solution to the "submodule membership problem" is an algorithm which can decide if an element m in $R^{(1\times NrColumns(M))}$ is contained in S or not. And exactly like the Gaussian (resp. Hermite) normal form when R is a field (resp. principal ideal ring), the row span of the resulting matrix B coincides with the row span S of `M`, and computing B is typically the first step of such an algorithm. (→ Appendix A)

5.5.23 BasisOfColumnModule (for matrices)

▷ `BasisOfColumnModule(M)` (operation)

Returns: a homalg matrix

Let R be the ring over which `M` is defined (R :=`HomalgRing( M )`) and S be the column span of `M`, i.e. the R-submodule of the free module $R^{(NrRows(M)\times 1)}$ spanned by the columns of `M`. A solution to the "submodule membership problem" is an algorithm which can decide if an element m in $R^{(NrRows(M)\times 1)}$ is contained in S or not. And exactly like the Gaussian (resp. Hermite) normal form when R is a field (resp. principal ideal ring), the column span of the resulting matrix B coincides with the column span S of `M`, and computing B is typically the first step of such an algorithm. (→ Appendix A)

5.5.24 DecideZeroRows (for pairs of matrices)

▷ `DecideZeroRows(A, B)` (operation)

Returns: a homalg matrix

Let `A` and `B` be matrices having the same number of columns and defined over the same ring R (:=`HomalgRing( A )`) and S be the row span of `B`, i.e. the R-submodule of the free module $R^{(1\times NrColumns(B))}$ spanned by the rows of `B`. The result is a matrix C having the same shape as `A`, for which the i-th row C^i is equivalent to the i-th row A^i of `A` modulo S, i.e. $C^i - A^i$ is an element of the row span S of `B`. Moreover, the row C^i is zero, if and only if the row A^i is an element of S. So `DecideZeroRows` decides which rows of `A` are zero modulo the rows of `B`. (→ Appendix A)

5.5.25 DecideZeroColumns (for pairs of matrices)

▷ `DecideZeroColumns(A, B)` (operation)

Returns: a homalg matrix

Let `A` and `B` be matrices having the same number of rows and defined over the same ring R (:=`HomalgRing( A )`) and S be the column span of `B`, i.e. the R-submodule of the free module $R^{(NrRows(B)\times 1)}$ spanned by the columns of `B`. The result is a matrix C having the same shape as `A`, for which the i-th column C_i is equivalent to the i-th column A_i of `A` modulo S, i.e. $C_i - A_i$ is an element of the column span S of `B`. Moreover, the column C_i is zero, if and only if the column A_i is an element of S. So `DecideZeroColumns` decides which columns of `A` are zero modulo the columns of `B`. (→ Appendix A)

5.5.26 SyzygiesGeneratorsOfRows (for matrices)

▷ SyzygiesGeneratorsOfRows(*M*) (operation)

Returns: a homalg matrix

Let R be the ring over which `M` is defined (R :=`HomalgRing( M )`). The matrix of row syzygies `SyzygiesGeneratorsOfRows( M )` is a matrix whose rows span the left kernel of `M`, i.e. the R-submodule of the free module $R^{(1\times NrRows(M))}$ consisting of all rows X satisfying $XM = 0$. (→ Appendix A)

5.5.27 SyzygiesGeneratorsOfColumns (for matrices)

▷ SyzygiesGeneratorsOfColumns(*M*) (operation)

Returns: a homalg matrix

Let R be the ring over which `M` is defined (R :=`HomalgRing( M )`). The matrix of column syzygies `SyzygiesGeneratorsOfColumns( M )` is a matrix whose columns span the right kernel of `M`, i.e. the R-submodule of the free module $R^{(NrColumns(M)\times 1)}$ consisting of all columns X satisfying $MX = 0$. (→ Appendix A)

5.5.28 SyzygiesGeneratorsOfRows (for pairs of matrices)

▷ SyzygiesGeneratorsOfRows(*M, M2*) (operation)

Returns: a homalg matrix

Let R be the ring over which `M` is defined (R :=`HomalgRing( M )`). The matrix of *relative* row syzygies `SyzygiesGeneratorsOfRows( M, M2 )` is a matrix whose rows span the left kernel of `M` modulo `M2`, i.e. the R-submodule of the free module $R^{(1\times NrRows(M))}$ consisting of all rows X satisfying $XM + YM2 = 0$ for some row $Y \in R^{(1\times NrRows(M2))}$. (→ Appendix A)

5.5.29 SyzygiesGeneratorsOfColumns (for pairs of matrices)

▷ SyzygiesGeneratorsOfColumns(*M, M2*) (operation)

Returns: a homalg matrix

Let R be the ring over which `M` is defined (R :=`HomalgRing( M )`). The matrix of *relative* column syzygies `SyzygiesGeneratorsOfColumns( M, M2 )` is a matrix whose columns span the right kernel of `M` modulo `M2`, i.e. the R-submodule of the free module $R^{(NrColumns(M)\times 1)}$ consisting of all columns X satisfying $MX + M2Y = 0$ for some column $Y \in R^{(NrColumns(M2)\times 1)}$. (→ Appendix A)

5.5.30 ReducedBasisOfRowModule (for matrices)

▷ ReducedBasisOfRowModule(*M*) (operation)

Returns: a homalg matrix

Like `BasisOfRowModule( M )` but where the matrix `SyzygiesGeneratorsOfRows( ReducedBasisOfRowModule( M ) )` contains no units. This can easily be achieved starting from B :=`BasisOfRowModule( M )` (and using `GetColumnIndependentUnitPositions` (5.5.18) applied to the matrix of row syzygies of B, etc). (→ Appendix A)

5.5.31 ReducedBasisOfColumnModule (for matrices)

▷ ReducedBasisOfColumnModule(*M*) (operation)

Returns: a homalg matrix

Like `BasisOfColumnModule( M )` but where the matrix `SyzygiesGeneratorsOfColumns( ReducedBasisOfColumnModule( M ) )` contains no units. This can easily be achieved starting from B :=`BasisOfColumnModule( M )` (and using `GetRowIndependentUnitPositions` (5.5.19) applied to the matrix of column syzygies of B, etc.). (→ Appendix A)

5.5.32 ReducedSyzygiesGeneratorsOfRows (for matrices)

▷ `ReducedSyzygiesGeneratorsOfRows(M)` (operation)

Returns: a homalg matrix

Like `SyzygiesGeneratorsOfRows( M )` but where the matrix `SyzygiesGeneratorsOfRows( ReducedSyzygiesGeneratorsOfRows( M ) )` contains no units. This can easily be achieved starting from C :=`SyzygiesGeneratorsOfRows( M )` (and using `GetColumnIndependentUnitPositions` (5.5.18) applied to the matrix of row syzygies of C, etc.). (→ Appendix A)

5.5.33 ReducedSyzygiesGeneratorsOfColumns (for matrices)

▷ `ReducedSyzygiesGeneratorsOfColumns(M)` (operation)

Returns: a homalg matrix

Like `SyzygiesGeneratorsOfColumns( M )` but where the matrix `SyzygiesGeneratorsOfColumns( ReducedSyzygiesGeneratorsOfColumns( M ) )` contains no units. This can easily be achieved starting from C :=`SyzygiesGeneratorsOfColumns( M )` (and using `GetRowIndependentUnitPositions` (5.5.19) applied to the matrix of column syzygies of C, etc.). (→ Appendix A)

5.5.34 BasisOfRowsCoeff (for matrices)

▷ `BasisOfRowsCoeff(M, T)` (operation)

Returns: a homalg matrix

Returns B :=`BasisOfRowModule( M )` and assigns the *void* matrix T (→ `HomalgVoidMatrix` (5.2.5)) such that $B = TM$. (→ Appendix A)

5.5.35 BasisOfColumnsCoeff (for matrices)

▷ `BasisOfColumnsCoeff(M, T)` (operation)

Returns: a homalg matrix

Returns B :=`BasisOfRowModule( M )` and assigns the *void* matrix T (→ `HomalgVoidMatrix` (5.2.5)) such that $B = MT$. (→ Appendix A)

5.5.36 DecideZeroRowsEffectively (for pairs of matrices)

▷ `DecideZeroRowsEffectively(A, B, T)` (operation)

Returns: a homalg matrix

Returns M :=`DecideZeroRows( A, B )` and assigns the *void* matrix T (→ `HomalgVoidMatrix` (5.2.5)) such that $M = A + TB$. (→ Appendix A)

5.5.37 DecideZeroColumnsEffectively (for pairs of matrices)

▷ `DecideZeroColumnsEffectively(A, B, T)` (operation)

Returns: a homalg matrix

Returns M :=`DecideZeroColumns(` A, B `)` and assigns the *void* matrix T (→ `HomalgVoidMatrix` (5.2.5)) such that $M = A + BT$. (→ Appendix A)

5.5.38 BasisOfRows (for matrices)

▷ `BasisOfRows(M)` (operation)

▷ `BasisOfRows(M, T)` (operation)

Returns: a homalg matrix

With one argument it is a synonym of `BasisOfRowModule` (5.5.22). with two arguments it is a synonym of `BasisOfRowsCoeff` (5.5.34).

5.5.39 BasisOfColumns (for matrices)

▷ `BasisOfColumns(M)` (operation)

▷ `BasisOfColumns(M, T)` (operation)

Returns: a homalg matrix

With one argument it is a synonym of `BasisOfColumnModule` (5.5.23). with two arguments it is a synonym of `BasisOfColumnsCoeff` (5.5.35).

5.5.40 DecideZero (for matrices and relations)

▷ `DecideZero(mat, rel)` (operation)

Returns: a homalg matrix

Code

```
InstallMethod( DecideZero,
        "for sets of ring relations",
        [ IsHomalgMatrix, IsHomalgRingRelations ],

  function( mat, rel )

    return DecideZero( mat,  MatrixOfRelations( rel ) );

end );
```

5.5.41 SyzygiesOfRows (for matrices)

▷ `SyzygiesOfRows(M)` (operation)

▷ `SyzygiesOfRows(M, M2)` (operation)

Returns: a homalg matrix

With one argument it is a synonym of `SyzygiesGeneratorsOfRows` (5.5.26). with two arguments it is a synonym of `SyzygiesGeneratorsOfRows` (5.5.28).

5.5.42 SyzygiesOfColumns (for matrices)

▷ `SyzygiesOfColumns(M)` (operation)
▷ `SyzygiesOfColumns(M, M2)` (operation)

Returns: a homalg matrix

With one argument it is a synonym of `SyzygiesGeneratorsOfColumns` (5.5.27). with two arguments it is a synonym of `SyzygiesGeneratorsOfColumns` (5.5.29).

5.5.43 ReducedSyzygiesOfRows (for matrices)

▷ `ReducedSyzygiesOfRows(M)` (operation)
▷ `ReducedSyzygiesOfRows(M, M2)` (operation)

Returns: a homalg matrix

With one argument it is a synonym of `ReducedSyzygiesGeneratorsOfRows` (5.5.32). With two arguments it calls `ReducedBasisOfRowModule( SyzygiesGeneratorsOfRows( M, M2 ) )`. (→ `ReducedBasisOfRowModule` (5.5.30) and `SyzygiesGeneratorsOfRows` (5.5.28))

5.5.44 ReducedSyzygiesOfColumns (for matrices)

▷ `ReducedSyzygiesOfColumns(M)` (operation)
▷ `ReducedSyzygiesOfColumns(M, M2)` (operation)

Returns: a homalg matrix

With one argument it is a synonym of `ReducedSyzygiesGeneratorsOfColumns` (5.5.33). With two arguments it calls `ReducedBasisOfColumnModule( SyzygiesGeneratorsOfColumns( M, M2 ) )`. (→ `ReducedBasisOfColumnModule` (5.5.31) and `SyzygiesGeneratorsOfColumns` (5.5.29))

5.5.45 RightDivide (for pairs of matrices)

▷ `RightDivide(B, A)` (operation)

Returns: a homalg matrix or fail

Let B and A be matrices having the same number of columns and defined over the same ring. The matrix `RightDivide( B, A )` is a particular solution of the inhomogeneous (one sided) linear system of equations $XA = B$ in case it is solvable. Otherwise `fail` is returned. The name `RightDivide` suggests "$X = BA^{-1}$". This generalizes `LeftInverse` (5.5.2) for which B becomes the identity matrix. (→ `SyzygiesGeneratorsOfRows` (5.5.26))

5.5.46 LeftDivide (for pairs of matrices)

▷ `LeftDivide(A, B)` (operation)

Returns: a homalg matrix or fail

Let A and B be matrices having the same number of rows and defined over the same ring. The matrix `LeftDivide( A, B )` is a particular solution of the inhomogeneous (one sided) linear system of equations $AX = B$ in case it is solvable. Otherwise `fail` is returned. The name `LeftDivide` suggests "$X = A^{-1}B$". This generalizes `RightInverse` (5.5.3) for which B becomes the identity matrix. (→ `SyzygiesGeneratorsOfColumns` (5.5.27))

5.5.47 RightDivide (for triples of matrices)

▷ RightDivide(*B, A, L*) (operation)

Returns: a homalg matrix or fail

Let B, A and L be matrices having the same number of columns and defined over the same ring. The matrix RightDivide(B, A, L) is a particular solution of the inhomogeneous (one sided) linear system of equations $XA+YL=B$ in case it is solvable (for some Y which is forgotten). Otherwise fail is returned. The name RightDivide suggests "$X=BA^{-1}$ modulo L". (Cf. [BR08, Subsection 3.1.1])

Code

```
InstallMethod( RightDivide,
        "for homalg matrices",
        [ IsHomalgMatrix, IsHomalgMatrix, IsHomalgMatrix ],

  function( B, A, L )        ## CAUTION: Do not use lazy evaluation here!!!
    local R, BL, ZA, AL, CA, IAL, ZB, CB, NF, X;

    R := HomalgRing( B );

    BL := BasisOfRows( L );

    ## first reduce A modulo L
    ZA := DecideZeroRows( A, BL );

    AL := UnionOfRows( ZA, BL );

    ## CA * AL = IAL
    CA := HomalgVoidMatrix( R );
    IAL := BasisOfRows( AL, CA );

    ## also reduce B modulo L
    ZB := DecideZeroRows( B, BL );

    ## knowing this will avoid computations
    IsOne( IAL );

    ## IsSpecialSubidentityMatrix( IAL );        ## does not increase performance

    ## NF = ZB + CB * IAL
    CB := HomalgVoidMatrix( R );
    NF := DecideZeroRowsEffectively( ZB, IAL, CB );

    ## NF <> 0
    if not IsZero( NF ) then
        return fail;
    fi;

    ## CD = -CB * CA => CD * A = B
    X := -CB * CertainColumns( CA, [ 1 .. NrRows( A ) ] );

    ## check assertion
    Assert( 5, IsZero( DecideZeroRows( X * A - B, BL ) ) );
```

```
    return X;

    ## technical: -CB * CA := (-CB) * CA and COLEM should take over
    ## since CB := -matrix

end );
```

5.5.48 LeftDivide (for triples of matrices)

▷ LeftDivide(A, B, L) (operation)

Returns: a homalg matrix or fail

Let A, B and L be matrices having the same number of columns and defined over the same ring. The matrix LeftDivide(A, B, L) is a particular solution of the inhomogeneous (one sided) linear system of equations $AX+LY=B$ in case it is solvable (for some Y which is forgotten). Otherwise fail is returned. The name LeftDivide suggests "$X = A^{-1}B$ modulo L". (Cf. [BR08, Subsection 3.1.1])

Code
```
InstallMethod( LeftDivide,
        "for homalg matrices",
        [ IsHomalgMatrix, IsHomalgMatrix, IsHomalgMatrix ],

  function( A, B, L )        ## CAUTION: Do not use lazy evaluation here!!!
    local R, BL, ZA, AL, CA, IAL, ZB, CB, NF, X;

    R := HomalgRing( B );

    BL := BasisOfColumns( L );

    ## first reduce A modulo L
    ZA := DecideZeroColumns( A, BL );

    AL := UnionOfColumns( ZA, BL );

    ## AL * CA = IAL
    CA := HomalgVoidMatrix( R );
    IAL := BasisOfColumns( AL, CA );

    ## also reduce B modulo L
    ZB := DecideZeroColumns( B, BL );

    ## knowing this will avoid computations
    IsOne( IAL );

    ## IsSpecialSubidentityMatrix( IAL );        ## does not increase performance

    ## NF = ZB + IAL * CB
    CB := HomalgVoidMatrix( R );
    NF := DecideZeroColumnsEffectively( ZB, IAL, CB );

    ## NF <> 0
    if not IsZero( NF ) then
```

```
        return fail;
    fi;

    ## CD = CA * -CB => A * CD = B
    X := CertainRows( CA, [ 1 .. NrColumns( A ) ] ) * -CB;

    ## check assertion
    Assert( 5, IsZero( DecideZeroColumns( A * X - B, BL ) ) );

    return X;

    ## technical: CA * -CB := CA * (-CB) and COLEM should take over since
    ## CB := -matrix

end );
```

5.5.49 GenerateSameRowModule (for pairs of matrices)

▷ `GenerateSameRowModule(M, N)` (operation)

Returns: `true` or `false`

Check if the row span of *M* and of *N* are identical or not (→ `RightDivide` (5.5.45)).

5.5.50 GenerateSameColumnModule (for pairs of matrices)

▷ `GenerateSameColumnModule(M, N)` (operation)

Returns: `true` or `false`

Check if the column span of *M* and of *N* are identical or not (→ `LeftDivide` (5.5.46)).

Chapter 6

Ring Relations

6.1 Ring Relations: Categories and Representations

6.1.1 IsHomalgRingRelations

▷ `IsHomalgRingRelations(rel)` (Category)

Returns: `true` or `false`

The GAP category of homalg ring relations.

6.1.2 IsHomalgRingRelationsAsGeneratorsOfLeftIdeal

▷ `IsHomalgRingRelationsAsGeneratorsOfLeftIdeal(rel)` (Category)

Returns: `true` or `false`

The GAP category of homalg ring relations as generators of a left ideal.

(It is a subcategory of the GAP category `IsHomalgRingRelations`.)

6.1.3 IsHomalgRingRelationsAsGeneratorsOfRightIdeal

▷ `IsHomalgRingRelationsAsGeneratorsOfRightIdeal(rel)` (Category)

Returns: `true` or `false`

The GAP category of homalg ring relations as generators of a right ideal.

(It is a subcategory of the GAP category `IsHomalgRingRelations`.)

6.1.4 IsRingRelationsRep

▷ `IsRingRelationsRep(rel)` (Representation)

Returns: `true` or `false`

The GAP representation of a finite set of relations of a homalg ring.

(It is a representation of the GAP category `IsHomalgRingRelations` (6.1.1))

6.2 Ring Relations: Constructors

6.3 Ring Relations: Properties

6.3.1 CanBeUsedToDecideZero

▷ `CanBeUsedToDecideZero(rel)` (property)

Returns: `true` or `false`

Check if the homalg set of relations *rel* can be used for normal form reductions.

(no method installed)

6.3.2 IsInjectivePresentation

▷ `IsInjectivePresentation(rel)` (property)

Returns: `true` or `false`

Check if the homalg set of relations *rel* has zero syzygies.

6.4 Ring Relations: Attributes

6.5 Ring Relations: Operations and Functions

Appendix A

The Basic Matrix Operations

These are the operations used to solve one-sided (in)homogeneous linear systems $XA = B$ resp. $AX = B$.

A.1 Main

- `BasisOfRowModule` (5.5.22)
- `BasisOfColumnModule` (5.5.23)
- `DecideZeroRows` (5.5.24)
- `DecideZeroColumns` (5.5.25)
- `SyzygiesGeneratorsOfRows` (5.5.26)
- `SyzygiesGeneratorsOfColumns` (5.5.27)

A.2 Effective

- `BasisOfRowsCoeff` (5.5.34)
- `BasisOfColumnsCoeff` (5.5.35)
- `DecideZeroRowsEffectively` (5.5.36)
- `DecideZeroColumnsEffectively` (5.5.37)

A.3 Relative

- `SyzygiesGeneratorsOfRows` (5.5.28)
- `SyzygiesGeneratorsOfColumns` (5.5.29)

A.4 Reduced

- `ReducedBasisOfRowModule` (5.5.30)
- `ReducedBasisOfColumnModule` (5.5.31)
- `ReducedSyzygiesGeneratorsOfRows` (5.5.32)
- `ReducedSyzygiesGeneratorsOfColumns` (5.5.33)

Appendix B

The Matrix Tool Operations

The functions listed below are components of the homalgTable object stored in the ring. They are only indirectly accessible through standard methods that invoke them.

B.1 The Tool Operations *without* a Fallback Method

There are matrix methods for which homalg needs a homalgTable entry for non-internal rings, as it cannot provide a suitable fallback. Below is the list of these homalgTable entries.

B.1.1 InitialMatrix (homalgTable entry for initial matrices)

▷ InitialMatrix(*C*) (function)

Returns: the Eval value of a homalg matrix *C*

Let R := HomalgRing(*C*) and RP := homalgTable(R). If the homalgTable component *RP*!.InitialMatrix is bound then the method Eval (C.4.1) resets the filter IsInitialMatrix and returns *RP*!.InitialMatrix(*C*).

B.1.2 InitialIdentityMatrix (homalgTable entry for initial identity matrices)

▷ InitialIdentityMatrix(*C*) (function)

Returns: the Eval value of a homalg matrix *C*

Let R := HomalgRing(*C*) and RP := homalgTable(R). If the homalgTable component *RP*!.InitialIdentityMatrix is bound then the method Eval (C.4.2) resets the filter IsInitialIdentityMatrix and returns *RP*!.InitialIdentityMatrix(*C*).

B.1.3 ZeroMatrix (homalgTable entry)

▷ ZeroMatrix(*C*) (function)

Returns: the Eval value of a homalg matrix *C*

Let R := HomalgRing(*C*) and RP := homalgTable(R). If the homalgTable component *RP*!.ZeroMatrix is bound then the method Eval (C.4.3) returns *RP*!.ZeroMatrix(*C*).

B.1.4 IdentityMatrix (homalgTable entry)

▷ `IdentityMatrix(C)` (function)

Returns: the `Eval` value of a homalg matrix C

Let R := `HomalgRing`(C) and RP := `homalgTable`(R). If the `homalgTable` component RP!.`IdentityMatrix` is bound then the method `Eval` (C.4.4) returns RP!.`IdentityMatrix`(C).

B.1.5 Involution (homalgTable entry)

▷ `Involution(M)` (function)

Returns: the `Eval` value of a homalg matrix C

Let R := `HomalgRing`(C) and RP := `homalgTable`(R). If the `homalgTable` component RP!.`Involution` is bound then the method `Eval` (C.4.7) returns RP!.`Involution` applied to the content of the attribute `EvalInvolution`(C) $= M$.

B.1.6 CertainRows (homalgTable entry)

▷ `CertainRows(M, plist)` (function)

Returns: the `Eval` value of a homalg matrix C

Let R := `HomalgRing`(C) and RP := `homalgTable`(R). If the `homalgTable` component RP!.`CertainRows` is bound then the method `Eval` (C.4.8) returns RP!.`CertainRows` applied to the content of the attribute `EvalCertainRows`(C) $= [M, plist]$.

B.1.7 CertainColumns (homalgTable entry)

▷ `CertainColumns(M, plist)` (function)

Returns: the `Eval` value of a homalg matrix C

Let R := `HomalgRing`(C) and RP := `homalgTable`(R). If the `homalgTable` component RP!.`CertainColumns` is bound then the method `Eval` (C.4.9) returns RP!.`CertainColumns` applied to the content of the attribute `EvalCertainColumns`(C) $= [M, plist]$.

B.1.8 UnionOfRows (homalgTable entry)

▷ `UnionOfRows(A, B)` (function)

Returns: the `Eval` value of a homalg matrix C

Let R := `HomalgRing`(C) and RP := `homalgTable`(R). If the `homalgTable` component RP!.`UnionOfRows` is bound then the method `Eval` (C.4.10) returns RP!.`UnionOfRows` applied to the content of the attribute `EvalUnionOfRows`(C) $= [A, B]$.

B.1.9 UnionOfColumns (homalgTable entry)

▷ `UnionOfColumns(A, B)` (function)

Returns: the `Eval` value of a homalg matrix C

Let R := `HomalgRing`(C) and RP := `homalgTable`(R). If the `homalgTable` component RP!.`UnionOfColumns` is bound then the method `Eval` (C.4.11) returns RP!.`UnionOfColumns` applied to the content of the attribute `EvalUnionOfColumns`(C) $= [A, B]$.

B.1.10 DiagMat (homalgTable entry)

▷ DiagMat(*e*) (function)

Returns: the Eval value of a homalg matrix *C*

Let R := HomalgRing(C) and RP := homalgTable(R). If the homalgTable component RP!.DiagMat is bound then the method Eval (C.4.12) returns RP!.DiagMat applied to the content of the attribute EvalDiagMat(C) $= e$.

B.1.11 KroneckerMat (homalgTable entry)

▷ KroneckerMat(*A*, *B*) (function)

Returns: the Eval value of a homalg matrix *C*

Let R := HomalgRing(C) and RP := homalgTable(R). If the homalgTable component RP!.KroneckerMat is bound then the method Eval (C.4.13) returns RP!.KroneckerMat applied to the content of the attribute EvalKroneckerMat(C) $= [A,B]$.

B.1.12 MulMat (homalgTable entry)

▷ MulMat(*a*, *A*) (function)

Returns: the Eval value of a homalg matrix *C*

Let R := HomalgRing(C) and RP := homalgTable(R). If the homalgTable component RP!.MulMat is bound then the method Eval (C.4.14) returns RP!.MulMat applied to the content of the attribute EvalMulMat(C) $= [a,A]$.

B.1.13 AddMat (homalgTable entry)

▷ AddMat(*A*, *B*) (function)

Returns: the Eval value of a homalg matrix *C*

Let R := HomalgRing(C) and RP := homalgTable(R). If the homalgTable component RP!.AddMat is bound then the method Eval (C.4.15) returns RP!.AddMat applied to the content of the attribute EvalAddMat(C) $= [A,B]$.

B.1.14 SubMat (homalgTable entry)

▷ SubMat(*A*, *B*) (function)

Returns: the Eval value of a homalg matrix *C*

Let R := HomalgRing(C) and RP := homalgTable(R). If the homalgTable component RP!.SubMat is bound then the method Eval (C.4.16) returns RP!.SubMat applied to the content of the attribute EvalSubMat(C) $= [A,B]$.

B.1.15 Compose (homalgTable entry)

▷ Compose(*A*, *B*) (function)

Returns: the Eval value of a homalg matrix *C*

Let R := HomalgRing(C) and RP := homalgTable(R). If the homalgTable component RP!.Compose is bound then the method Eval (C.4.17) returns RP!.Compose applied to the content of the attribute EvalCompose(C) $= [A,B]$.

B.1.16 IsZeroMatrix (homalgTable entry)

▷ IsZeroMatrix(*M*) (function)

Returns: true or false

Let R := HomalgRing(M) and RP := homalgTable(R). If the homalgTable component RP!.IsZeroMatrix is bound then the standard method for the property IsZero (5.3.1) shown below returns RP!.IsZeroMatrix(M).

Code
```
InstallMethod( IsZero,
        "for homalg matrices",
        [ IsHomalgMatrix ],

  function( M )
    local R, RP;

    R := HomalgRing( M );

    RP := homalgTable( R );

    if IsBound(RP!.IsZeroMatrix) then
        ## CAUTION: the external system must be able
        ## to check zero modulo possible ring relations!

        return RP!.IsZeroMatrix( M ); ## with this, \= can fall back to IsZero
    fi;

    #=====# the fallback method #=====#

    ## from the GAP4 documentation: ?Zero
    ## `ZeroSameMutability( <obj> )' is equivalent to `0 * <obj>'.

    return M = 0 * M; ## hence, by default, IsZero falls back to \= (see below)

end );
```

B.1.17 NrRows (homalgTable entry)

▷ NrRows(*C*) (function)

Returns: a nonnegative integer

Let R := HomalgRing(C) and RP := homalgTable(R). If the homalgTable component RP!.NrRows is bound then the standard method for the attribute NrRows (5.4.1) shown below returns RP!.NrRows(C).

Code
```
InstallMethod( NrRows,
        "for homalg matrices",
        [ IsHomalgMatrix ],

  function( C )
    local R, RP;

    R := HomalgRing( C );
```

```
    RP := homalgTable( R );

    if IsBound(RP!.NrRows) then
        return RP!.NrRows( C );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called NrRows ",
               "in the homalgTable of the non-internal ring\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#

    return Length( Eval( C )!.matrix );

end );
```

B.1.18 NrColumns (homalgTable entry)

▷ NrColumns(*C*) (function)

Returns: a nonnegative integer

Let $R :=$ HomalgRing(C) and $RP :=$ homalgTable(R). If the homalgTable component RP!.NrColumns is bound then the standard method for the attribute NrColumns (5.4.2) shown below returns RP!.NrColumns(C).

```
InstallMethod( NrColumns,
        "for homalg matrices",
        [ IsHomalgMatrix ],

  function( C )
    local R, RP;

    R := HomalgRing( C );

    RP := homalgTable( R );

    if IsBound(RP!.NrColumns) then
        return RP!.NrColumns( C );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called NrColumns ",
               "in the homalgTable of the non-internal ring\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#

    return Length( Eval( C )!.matrix[ 1 ] );

end );
```

B.1.19 Determinant (homalgTable entry)

▷ Determinant(*C*) (function)

Returns: a ring element

Let R := HomalgRing(C) and RP := homalgTable(R). If the homalgTable component RP!.Determinant is bound then the standard method for the attribute DeterminantMat (5.4.3) shown below returns RP!.Determinant(C).

Code
```
InstallMethod( DeterminantMat,
        "for homalg matrices",
        [ IsHomalgMatrix ],

  function( C )
    local R, RP;

    R := HomalgRing( C );

    RP := homalgTable( R );

    if NrRows( C ) <> NrColumns( C ) then
        Error( "the matrix is not a square matrix\n" );
    fi;

    if IsEmptyMatrix( C ) then
        return One( R );
    elif IsZero( C ) then
        return Zero( R );
    fi;

    if IsBound(RP!.Determinant) then
        return RingElementConstructor( R )( RP!.Determinant( C ), R );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called Determinant ",
               "in the homalgTable of the non-internal ring\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#

    return Determinant( Eval( C )!.matrix );

end );


InstallMethod( Determinant,
        "for homalg matrices",
        [ IsHomalgMatrix ],

  function( C )

    return DeterminantMat( C );
```

```
end );
```

B.2 The Tool Operations with a Fallback Method

These are the methods for which it is recommended for performance reasons to have a `homalgTable` entry for non-internal rings. homalg only provides a generic fallback method.

B.2.1 AreEqualMatrices (homalgTable entry)

▷ `AreEqualMatrices(M1, M2)` (function)

Returns: `true` or `false`

Let R := `HomalgRing(M1)` and RP := `homalgTable(R)`. If the `homalgTable` component RP!.`AreEqualMatrices` is bound then the standard method for the operation \= (5.5.17) shown below returns RP!.`AreEqualMatrices(M1,M2)`.

```
InstallMethod( \=,
        "for homalg comparable matrices",
        [ IsHomalgMatrix, IsHomalgMatrix ],

  function( M1, M2 )
    local R, RP, are_equal;

    ## do not touch mutable matrices
    if not ( IsMutable( M1 ) or IsMutable( M2 ) ) then

        if IsBound( M1!.AreEqual ) then
            are_equal := _ElmWPObj_ForHomalg( M1!.AreEqual, M2, fail );
            if are_equal <> fail then
                return are_equal;
            fi;
        else
            M1!.AreEqual :=
              ContainerForWeakPointers(
                      TheTypeContainerForWeakPointersOnComputedValues,
                      [ "operation", "AreEqual" ] );
        fi;

        if IsBound( M2!.AreEqual ) then
            are_equal := _ElmWPObj_ForHomalg( M2!.AreEqual, M1, fail );
            if are_equal <> fail then
                return are_equal;
            fi;
        fi;
        ## do not store things symmetrically below to ‘‘save’’ memory

    fi;

    R := HomalgRing( M1 );

    RP := homalgTable( R );
```

```
    if IsBound(RP!.AreEqualMatrices) then
        ## CAUTION: the external system must be able to check equality
        ## modulo possible ring relations (known to the external system)!
        are_equal := RP!.AreEqualMatrices( M1, M2 );
    elif IsBound(RP!.Equal) then
        ## CAUTION: the external system must be able to check equality
        ## modulo possible ring relations (known to the external system)!
        are_equal := RP!.Equal( M1, M2 );
    elif IsBound(RP!.IsZeroMatrix) then   ## ensuring this avoids infinite loops
        are_equal := IsZero( M1 - M2 );
    fi;

    if IsBound( are_equal ) then

        ## do not touch mutable matrices
        if not ( IsMutable( M1 ) or IsMutable( M2 ) ) then

            if are_equal then
                MatchPropertiesAndAttributes( M1, M2,
                        LIMAT.intrinsic_properties,
                        LIMAT.intrinsic_attributes,
                        LIMAT.intrinsic_components
                        );
            fi;

            ## do not store things symmetrically to ``save'' memory
            _AddTwoElmWPObj_ForHomalg( M1!.AreEqual, M2, are_equal );

        fi;

        return are_equal;
    fi;

    TryNextMethod( );

end );
```

B.2.2 IsIdentityMatrix (homalgTable entry)

▷ `IsIdentityMatrix(M)` (function)

Returns: `true` or `false`

Let $R :=$ `HomalgRing`(M) and $RP :=$ `homalgTable`(R). If the `homalgTable` component RP!.`IsIdentityMatrix` is bound then the standard method for the property `IsOne` (5.3.2) shown below returns RP!.`IsIdentityMatrix`(M).

Code
```
InstallMethod( IsOne,
        "for homalg matrices",
        [ IsHomalgMatrix ],

  function( M )
    local R, RP;
```

```
    if NrRows( M ) <> NrColumns( M ) then
        return false;
    fi;

    R := HomalgRing( M );

    RP := homalgTable( R );

    if IsBound(RP!.IsIdentityMatrix) then
        return RP!.IsIdentityMatrix( M );
    fi;

    #=====# the fallback method #=====#

    return M = HomalgIdentityMatrix( NrRows( M ), HomalgRing( M ) );

end );
```

B.2.3 IsDiagonalMatrix (homalgTable entry)

▷ `IsDiagonalMatrix(M)` (function)

Returns: `true` or `false`

Let R := `HomalgRing`(M) and RP := `homalgTable`(R). If the `homalgTable` component RP!.`IsDiagonalMatrix` is bound then the standard method for the property `IsDiagonalMatrix` (5.3.13) shown below returns RP!.`IsDiagonalMatrix`(M).

Code

```
InstallMethod( IsDiagonalMatrix,
        "for homalg matrices",
        [ IsHomalgMatrix ],

  function( M )
    local R, RP, diag;

    R := HomalgRing( M );

    RP := homalgTable( R );

    if IsBound(RP!.IsDiagonalMatrix) then
        return RP!.IsDiagonalMatrix( M );
    fi;

    #=====# the fallback method #=====#

    diag := DiagonalEntries( M );

    return M = HomalgDiagonalMatrix( diag, NrRows( M ), NrColumns( M ), R );

end );
```

B.2.4 ZeroRows (homalgTable entry)

▷ ZeroRows(*C*) (function)

Returns: a (possibly empty) list of positive integers

Let R := HomalgRing(C) and RP := homalgTable(R). If the homalgTable component RP!.ZeroRows is bound then the standard method of the attribute ZeroRows (5.4.4) shown below returns RP!.ZeroRows(C).

Code

```
InstallMethod( ZeroRows,
        "for homalg matrices",
        [ IsHomalgMatrix ],

  function( C )
    local R, RP, z;

    R := HomalgRing( C );

    RP := homalgTable( R );

    if IsBound(RP!.ZeroRows) then
        return RP!.ZeroRows( C );
    fi;

    #=====# the fallback method #=====#

    z := HomalgZeroMatrix( 1, NrColumns( C ), R );

    return Filtered( [ 1 .. NrRows( C ) ], a -> CertainRows( C, [ a ] ) = z );

end );
```

B.2.5 ZeroColumns (homalgTable entry)

▷ ZeroColumns(*C*) (function)

Returns: a (possibly empty) list of positive integers

Let R := HomalgRing(C) and RP := homalgTable(R). If the homalgTable component RP!.ZeroColumns is bound then the standard method of the attribute ZeroColumns (5.4.5) shown below returns RP!.ZeroColumns(C).

Code

```
InstallMethod( ZeroColumns,
        "for homalg matrices",
        [ IsHomalgMatrix ],

  function( C )
    local R, RP, z;

    R := HomalgRing( C );

    RP := homalgTable( R );

    if IsBound(RP!.ZeroColumns) then
```

```
        return RP!.ZeroColumns( C );
    fi;

    #=====# the fallback method #=====#

    z := HomalgZeroMatrix( NrRows( C ), 1, R );

    return Filtered( [ 1 .. NrColumns( C ) ], a -> CertainColumns( C, [ a ] ) = z );

end );
```

B.2.6 GetColumnIndependentUnitPositions (homalgTable entry)

▷ GetColumnIndependentUnitPositions(*M, poslist*) (function)

Returns: a (possibly empty) list of pairs of positive integers

Let R := HomalgRing(M) and RP := homalgTable(R). If the homalgTable component RP!.GetColumnIndependentUnitPositions is bound then the standard method of the operation GetColumnIndependentUnitPositions (5.5.18) shown below returns RP!.GetColumnIndependentUnitPositions(M,*poslist*).

Code

```
InstallMethod( GetColumnIndependentUnitPositions,
        "for homalg matrices",
        [ IsHomalgMatrix, IsHomogeneousList ],

  function( M, poslist )
    local cache, R, RP, rest, pos, i, j, k;

    if IsBound( M!.GetColumnIndependentUnitPositions ) then
        cache := M!.GetColumnIndependentUnitPositions;
        if IsBound( cache.(String( poslist )) ) then
            return cache.(String( poslist ));
        fi;
    else
        cache := rec( );
        M!.GetColumnIndependentUnitPositions := cache;
    fi;

    R := HomalgRing( M );

    RP := homalgTable( R );

    if IsBound(RP!.GetColumnIndependentUnitPositions) then
        pos := RP!.GetColumnIndependentUnitPositions( M, poslist );
        if pos <> [ ] then
            SetIsZero( M, false );
        fi;
        cache.(String( poslist )) := pos;
        return pos;
    fi;

    #=====# the fallback method #=====#
```

```
    rest := [ 1 .. NrColumns( M ) ];

    pos := [ ];

    for i in [ 1 .. NrRows( M ) ] do
        for k in Reversed( rest ) do
            if not [ i, k ] in poslist and
               IsUnit( R, MatElm( M, i, k ) ) then
                Add( pos, [ i, k ] );
                rest := Filtered( rest,
                                a -> IsZero( MatElm( M, i, a ) ) );
                break;
            fi;
        od;
    od;

    if pos <> [ ] then
        SetIsZero( M, false );
    fi;

    cache.(String( poslist )) := pos;

    return pos;

end );
```

B.2.7 GetRowIndependentUnitPositions (homalgTable entry)

▷ GetRowIndependentUnitPositions(*M, poslist*) (function)

Returns: a (possibly empty) list of pairs of positive integers

Let R := HomalgRing(M) and RP := homalgTable(R). If the homalgTable component RP!.GetRowIndependentUnitPositions is bound then the standard method of the operation GetRowIndependentUnitPositions (5.5.19) shown below returns RP!.GetRowIndependentUnitPositions(M,*poslist*).

Code
```
InstallMethod( GetRowIndependentUnitPositions,
        "for homalg matrices",
        [ IsHomalgMatrix, IsHomogeneousList ],

  function( M, poslist )
    local cache, R, RP, rest, pos, j, i, k;

    if IsBound( M!.GetRowIndependentUnitPositions ) then
        cache := M!.GetRowIndependentUnitPositions;
        if IsBound( cache.(String( poslist )) ) then
            return cache.(String( poslist ));
        fi;
    else
        cache := rec( );
        M!.GetRowIndependentUnitPositions := cache;
```

```
    fi;

    R := HomalgRing( M );

    RP := homalgTable( R );

    if IsBound(RP!.GetRowIndependentUnitPositions) then
        pos := RP!.GetRowIndependentUnitPositions( M, poslist );
        if pos <> [ ] then
            SetIsZero( M, false );
        fi;
        cache.( String( poslist ) ) := pos;
        return pos;
    fi;

    #=====# the fallback method #=====#

    rest := [ 1 .. NrRows( M ) ];

    pos := [ ];

    for j in [ 1 .. NrColumns( M ) ] do
        for k in Reversed( rest ) do
            if not [ j, k ] in poslist and
               IsUnit( R, MatElm( M, k, j ) ) then
                Add( pos, [ j, k ] );
                rest := Filtered( rest,
                                a -> IsZero( MatElm( M, a, j ) ) );
                break;
            fi;
        od;
    od;

    if pos <> [ ] then
        SetIsZero( M, false );
    fi;

    cache.( String( poslist ) ) := pos;

    return pos;

end );
```

B.2.8 GetUnitPosition (homalgTable entry)

▷ `GetUnitPosition(M, poslist)` (function)

Returns: a (possibly empty) list of pairs of positive integers

Let $R :=$ `HomalgRing`(M) and $RP :=$ `homalgTable`(R). If the `homalgTable` component RP!.`GetUnitPosition` is bound then the standard method of the operation `GetUnitPosition` (5.5.20) shown below returns RP!.`GetUnitPosition`$(M, poslist)$.

Code
```
InstallMethod( GetUnitPosition,
```

```
        "for homalg matrices",
        [ IsHomalgMatrix, IsHomogeneousList ],

  function( M, poslist )
    local R, RP, pos, m, n, i, j;

    R := HomalgRing( M );

    RP := homalgTable( R );

    if IsBound(RP!.GetUnitPosition) then
        pos := RP!.GetUnitPosition( M, poslist );
        if IsList( pos ) and IsPosInt( pos[1] ) and IsPosInt( pos[2] ) then
            SetIsZero( M, false );
        fi;
        return pos;
    fi;

    #=====# the fallback method #=====#

    m := NrRows( M );
    n := NrColumns( M );

    for i in [ 1 .. m ] do
        for j in [ 1 .. n ] do
            if not [ i, j ] in poslist and not j in poslist and
               IsUnit( R, MatElm( M, i, j ) ) then
                SetIsZero( M, false );
                return [ i, j ];
            fi;
        od;
    od;

    return fail;

end );
```

B.2.9 PositionOfFirstNonZeroEntryPerRow (homalgTable entry)

▷ PositionOfFirstNonZeroEntryPerRow(*M, poslist*) (function)

Returns: a list of nonnegative integers

Let R := HomalgRing(M) and RP := homalgTable(R). If the homalgTable component RP!.PositionOfFirstNonZeroEntryPerRow is bound then the standard method of the attribute PositionOfFirstNonZeroEntryPerRow (5.4.8) shown below returns RP!.PositionOfFirstNonZeroEntryPerRow(M).

Code
```
InstallMethod( PositionOfFirstNonZeroEntryPerRow,
        "for homalg matrices",
        [ IsHomalgMatrix ],

  function( M )
```

```
    local R, RP, pos, entries, r, c, i, k, j;

    R := HomalgRing( M );

    RP := homalgTable( R );

    if IsBound(RP!.PositionOfFirstNonZeroEntryPerRow) then
        return RP!.PositionOfFirstNonZeroEntryPerRow( M );
    elif IsBound(RP!.PositionOfFirstNonZeroEntryPerColumn) then
        return PositionOfFirstNonZeroEntryPerColumn( Involution( M ) );
    fi;

    #=====# the fallback method #=====#

    entries := EntriesOfHomalgMatrix( M );

    r := NrRows( M );
    c := NrColumns( M );

    pos := ListWithIdenticalEntries( r, 0 );

    for i in [ 1 .. r ] do
        k := (i - 1) * c;
        for j in [ 1 .. c ] do
            if not IsZero( entries[k + j] ) then
                pos[i] := j;
                break;
            fi;
        od;
    od;

    return pos;

end );
```

B.2.10 PositionOfFirstNonZeroEntryPerColumn (homalgTable entry)

▷ PositionOfFirstNonZeroEntryPerColumn(*M, poslist*) (function)

Returns: a list of nonnegative integers

Let $R :=$ HomalgRing(M) and $RP :=$ homalgTable(R). If the homalgTable component RP!.PositionOfFirstNonZeroEntryPerColumn is bound then the standard method of the attribute PositionOfFirstNonZeroEntryPerColumn (5.4.9) shown below returns RP!.PositionOfFirstNonZeroEntryPerColumn(M).

Code
```
InstallMethod( PositionOfFirstNonZeroEntryPerColumn,
        "for homalg matrices",
        [ IsHomalgMatrix ],

  function( M )
    local R, RP, pos, entries, r, c, j, i, k;

```

```
    R := HomalgRing( M );

    RP := homalgTable( R );

    if IsBound(RP!.PositionOfFirstNonZeroEntryPerColumn) then
        return RP!.PositionOfFirstNonZeroEntryPerColumn( M );
    elif IsBound(RP!.PositionOfFirstNonZeroEntryPerRow) then
        return PositionOfFirstNonZeroEntryPerRow( Involution( M ) );
    fi;

    #=====# the fallback method #=====#

    entries := EntriesOfHomalgMatrix( M );

    r := NrRows( M );
    c := NrColumns( M );

    pos := ListWithIdenticalEntries( c, 0 );

    for j in [ 1 .. c ] do
        for i in [ 1 .. r ] do
            k := (i - 1) * c;
            if not IsZero( entries[k + j] ) then
                pos[j] := i;
                break;
            fi;
        od;
    od;

    return pos;

end );
```

Appendix C

Logic Subpackages

C.1 LIRNG: Logical Implications for Rings

C.2 LIMAP: Logical Implications for Ring Maps

C.3 LIMAT: Logical Implications for Matrices

C.4 COLEM: Clever Operations for Lazy Evaluated Matrices

Most of the matrix tool operations listed in Appendix B.1 which return a new matrix are lazy evaluated. The value of a homalg matrix is stored in the attribute `Eval`. Below is the list of the installed methods for the attribute `Eval`.

C.4.1 Eval (for matrices created with HomalgInitialMatrix)

▷ `Eval(C)` (method)

Returns: the `Eval` value of a homalg matrix *C*

In case the matrix *C* was created using `HomalgInitialMatrix` (5.2.1) then the filter `IsInitialMatrix` for *C* is set to true and the `homalgTable` function (→ `InitialMatrix` (B.1.1)) will be used to set the attribute `Eval` and resets the filter `IsInitialMatrix`.

Code

```
InstallMethod( Eval,
        "for homalg matrices (IsInitialMatrix)",
        [ IsHomalgMatrix and IsInitialMatrix and
          HasNrRows and HasNrColumns ],

  function( C )
    local R, RP, z, zz;

    R := HomalgRing( C );

    RP := homalgTable( R );

    if IsBound( RP!.InitialMatrix ) then
        ResetFilterObj( C, IsInitialMatrix );
        return RP!.InitialMatrix( C );
```

```
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called InitialMatrix in the ",
               "homalgTable to evaluate a non-internal initial matrix\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#

    z := Zero( HomalgRing( C ) );

    ResetFilterObj( C, IsInitialMatrix );

    zz := ListWithIdenticalEntries( NrColumns( C ), z );

    return homalgInternalMatrixHull(
                   List( [ 1 .. NrRows( C ) ], i -> ShallowCopy( zz ) ) );

end );
```

C.4.2 Eval (for matrices created with HomalgInitialIdentityMatrix)

▷ Eval(*C*) (method)

Returns: the Eval value of a homalg matrix *C*

In case the matrix *C* was created using HomalgInitialIdentityMatrix (5.2.2) then the filter IsInitialIdentityMatrix for *C* is set to true and the homalgTable function (→ InitialIdentityMatrix (B.1.2)) will be used to set the attribute Eval and resets the filter IsInitialIdentityMatrix.

Code

```
InstallMethod( Eval,
        "for homalg matrices (IsInitialIdentityMatrix)",
        [ IsHomalgMatrix and IsInitialIdentityMatrix and
          HasNrRows and HasNrColumns ],

  function( C )
    local R, RP, o, z, zz, id;

    R := HomalgRing( C );

    RP := homalgTable( R );

    if IsBound( RP!.InitialIdentityMatrix ) then
        ResetFilterObj( C, IsInitialIdentityMatrix );
        return RP!.InitialIdentityMatrix( C );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called InitialIdentityMatrix in the ",
               "homalgTable to evaluate a non-internal initial identity matrix\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#
```

```
    z := Zero( HomalgRing( C ) );
    o := One( HomalgRing( C ) );

    ResetFilterObj( C, IsInitialIdentityMatrix );

    zz := ListWithIdenticalEntries( NrColumns( C ), z );

    id := List( [ 1 .. NrRows( C ) ],
                function(i)
                  local z;
                  z := ShallowCopy( zz ); z[i] := o; return z;
                end );

    return homalgInternalMatrixHull( id );

end );
```

C.4.3 Eval (for matrices created with HomalgZeroMatrix)

▷ `Eval(C)` (method)

Returns: the `Eval` value of a homalg matrix *C*

In case the matrix *C* was created using `HomalgZeroMatrix` (5.2.3) then the filter `IsZeroMatrix` for *C* is set to true and the `homalgTable` function (→ `ZeroMatrix` (B.1.3)) will be used to set the attribute `Eval`.

Code

```
InstallMethod( Eval,
        "for homalg matrices (IsZero)",
        [ IsHomalgMatrix and IsZero and HasNrRows and HasNrColumns ], 20,

  function( C )
    local R, RP, z;

    R := HomalgRing( C );

    RP := homalgTable( R );

    if ( NrRows( C ) = 0 or NrColumns( C ) = 0 ) and
       not ( IsBound( R!.SafeToEvaluateEmptyMatrices ) and
             R!.SafeToEvaluateEmptyMatrices = true ) then
        Info( InfoWarning, 1, "\033[01m\033[5;31;47m",
              "an empty matrix is about to get evaluated!",
              "\033[0m" );
    fi;

    if IsBound( RP!.ZeroMatrix ) then
        return RP!.ZeroMatrix( C );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called ZeroMatrix ",
               "homalgTable to evaluate a non-internal zero matrix\n" );
```

```
    fi;

    #=====# can only work for homalg internal matrices #=====#

    z := Zero( HomalgRing( C ) );

    ## copying the rows saves memory;
    ## we assume that the entries are never modified!!!
    return homalgInternalMatrixHull(
                   ListWithIdenticalEntries( NrRows( C ),
                           ListWithIdenticalEntries( NrColumns( C ), z ) ) );

end );
```

C.4.4 Eval (for matrices created with HomalgIdentityMatrix)

▷ Eval(*C*) (method)

Returns: the Eval value of a homalg matrix *C*

In case the matrix *C* was created using HomalgIdentityMatrix (5.2.4) then the filter IsOne for *C* is set to true and the homalgTable function (→ IdentityMatrix (B.1.4)) will be used to set the attribute Eval.

Code
```
InstallMethod( Eval,
        "for homalg matrices (IsOne)",
        [ IsHomalgMatrix and IsOne and HasNrRows and HasNrColumns ], 10,

  function( C )
    local R, id, RP, o, z, zz;

    R := HomalgRing( C );

    if IsBound( R!.IdentityMatrices ) then
        id := ElmWPObj( R!.IdentityMatrices!.weak_pointers, NrColumns( C ) );
        if id <> fail then
            R!.IdentityMatrices!.cache_hits := R!.IdentityMatrices!.cache_hits + 1;
            return id;
        fi;
        ## we do not count cache_misses as it is equivalent to counter
    fi;

    RP := homalgTable( R );

    if IsBound( RP!.IdentityMatrix ) then
        id := RP!.IdentityMatrix( C );
        SetElmWPObj( R!.IdentityMatrices!.weak_pointers, NrColumns( C ), id );
        R!.IdentityMatrices!.counter := R!.IdentityMatrices!.counter + 1;
        return id;
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called IdentityMatrix ",
               "homalgTable to evaluate a non-internal identity matrix\n" );
```

```
    fi;

    #=====# can only work for homalg internal matrices #=====#

    z := Zero( HomalgRing( C ) );
    o := One( HomalgRing( C ) );

    zz := ListWithIdenticalEntries( NrColumns( C ), z );

    id := List( [ 1 .. NrRows( C ) ],
                function(i)
                  local z;
                  z := ShallowCopy( zz ); z[i] := o; return z;
                end );

    id := homalgInternalMatrixHull( id );

    SetElmWPObj( R!.IdentityMatrices!.weak_pointers, NrColumns( C ), id );

    return id;

end );
```

C.4.5 Eval (for matrices created with LeftInverseLazy)

▷ Eval(*LI*) (method)

Returns: see below

In case the matrix *LI* was created using LeftInverseLazy (5.5.4) then the filter HasEvalLeftInverse for *LI* is set to true and the method listed below will be used to set the attribute Eval. (→ LeftInverse (5.5.2))

Code

```
InstallMethod( Eval,
        "for homalg matrices",
        [ IsHomalgMatrix and HasEvalLeftInverse ],

  function( LI )
    local left_inv;

    left_inv := LeftInverse( EvalLeftInverse( LI ) );

    if IsBool( left_inv ) then
        return false;
    fi;

    return Eval( left_inv );

end );
```

C.4.6 Eval (for matrices created with RightInverseLazy)

▷ Eval(*RI*) (method)

Returns: see below

In case the matrix *RI* was created using RightInverseLazy (5.5.5) then the filter HasEvalRightInverse for *RI* is set to true and the method listed below will be used to set the attribute Eval. (→ RightInverse (5.5.3))

Code
```
InstallMethod( Eval,
        "for homalg matrices",
        [ IsHomalgMatrix and HasEvalRightInverse ],

  function( RI )
    local right_inv;

    right_inv := RightInverse( EvalRightInverse( RI ) );

    if IsBool( right_inv ) then
        return false;
    fi;

    return Eval( right_inv );

end );
```

C.4.7 Eval (for matrices created with Involution)

▷ Eval(*C*) (method)

Returns: the Eval value of a homalg matrix *C*

In case the matrix was created using Involution (5.5.6) then the filter HasEvalInvolution for *C* is set to true and the homalgTable function Involution (B.1.5) will be used to set the attribute Eval.

Code
```
InstallMethod( Eval,
        "for homalg matrices (HasEvalInvolution)",
        [ IsHomalgMatrix and HasEvalInvolution ],

  function( C )
    local R, RP, M;

    R := HomalgRing( C );

    RP := homalgTable( R );

    M :=  EvalInvolution( C );

    if IsBound(RP!.Involution) then
        return RP!.Involution( M );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called Involution ",
```

```
                "in the homalgTable of the non-internal ring\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#

    return homalgInternalMatrixHull( TransposedMat( Eval( M )!.matrix ) );

end );
```

C.4.8 Eval (for matrices created with CertainRows)

▷ `Eval(C)` (method)

Returns: the `Eval` value of a homalg matrix *C*

In case the matrix was created using `CertainRows` (5.5.7) then the filter `HasEvalCertainRows` for *C* is set to true and the `homalgTable` function `CertainRows` (B.1.6) will be used to set the attribute `Eval`.

Code
```
InstallMethod( Eval,
        "for homalg matrices (HasEvalCertainRows)",
        [ IsHomalgMatrix and HasEvalCertainRows ],

  function( C )
    local R, RP, e, M, plist;

    R := HomalgRing( C );

    RP := homalgTable( R );

    e :=  EvalCertainRows( C );

    M := e[1];
    plist := e[2];

    if IsBound(RP!.CertainRows) then
        return RP!.CertainRows( M, plist );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called CertainRows ",
               "in the homalgTable of the non-internal ring\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#

    return homalgInternalMatrixHull( Eval( M )!.matrix{ plist } );

end );
```

C.4.9 Eval (for matrices created with CertainColumns)

▷ Eval(*C*) (method)

Returns: the Eval value of a homalg matrix *C*

In case the matrix was created using CertainColumns (5.5.8) then the filter HasEvalCertainColumns for *C* is set to true and the homalgTable function CertainColumns (B.1.7) will be used to set the attribute Eval.

Code
```
InstallMethod( Eval,
        "for homalg matrices (HasEvalCertainColumns)",
        [ IsHomalgMatrix and HasEvalCertainColumns ],

  function( C )
    local R, RP, e, M, plist;

    R := HomalgRing( C );

    RP := homalgTable( R );

    e :=  EvalCertainColumns( C );

    M := e[1];
    plist := e[2];

    if IsBound(RP!.CertainColumns) then
        return RP!.CertainColumns( M, plist );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called CertainColumns ",
               "in the homalgTable of the non-internal ring\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#

    return homalgInternalMatrixHull(
                   Eval( M )!.matrix{[ 1 .. NrRows( M ) ]}{plist} );

end );
```

C.4.10 Eval (for matrices created with UnionOfRows)

▷ Eval(*C*) (method)

Returns: the Eval value of a homalg matrix *C*

In case the matrix was created using UnionOfRows (5.5.9) then the filter HasEvalUnionOfRows for *C* is set to true and the homalgTable function UnionOfRows (B.1.8) will be used to set the attribute Eval.

Code
```
InstallMethod( Eval,
        "for homalg matrices (HasEvalUnionOfRows)",
        [ IsHomalgMatrix and HasEvalUnionOfRows ],
```

```
  function( C )
    local R, RP, e, A, B, U;

    R := HomalgRing( C );

    RP := homalgTable( R );

    e :=  EvalUnionOfRows( C );

    A := e[1];
    B := e[2];

    if IsBound(RP!.UnionOfRows) then
        return RP!.UnionOfRows( A, B );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called UnionOfRows ",
               "in the homalgTable of the non-internal ring\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#

    U := ShallowCopy( Eval( A )!.matrix );

    U{ [ NrRows( A ) + 1 .. NrRows( A ) + NrRows( B ) ] } := Eval( B )!.matrix;

    return homalgInternalMatrixHull( U );

end );
```

C.4.11 Eval (for matrices created with UnionOfColumns)

▷ `Eval(C)` (method)

Returns: the `Eval` value of a homalg matrix *C*

In case the matrix was created using `UnionOfColumns` (5.5.10) then the filter `HasEvalUnionOfColumns` for *C* is set to true and the `homalgTable` function `UnionOfColumns` (B.1.9) will be used to set the attribute `Eval`.

Code

```
InstallMethod( Eval,
        "for homalg matrices (HasEvalUnionOfColumns)",
        [ IsHomalgMatrix and HasEvalUnionOfColumns ],

  function( C )
    local R, RP, e, A, B, U;

    R := HomalgRing( C );

    RP := homalgTable( R );

    e :=  EvalUnionOfColumns( C );
```

```
    A := e[1];
    B := e[2];

    if IsBound(RP!.UnionOfColumns) then
        return RP!.UnionOfColumns( A, B );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called UnionOfColumns ",
               "in the homalgTable of the non-internal ring\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#

    U := List( Eval( A )!.matrix, ShallowCopy );

    U{ [ 1 .. NrRows( A ) ] }
      { [ NrColumns( A ) + 1 .. NrColumns( A ) + NrColumns( B ) ] }
      := Eval( B )!.matrix;

    return homalgInternalMatrixHull( U );

end );
```

C.4.12 Eval (for matrices created with DiagMat)

▷ Eval(*C*) (method)

Returns: the Eval value of a homalg matrix *C*

In case the matrix was created using DiagMat (5.5.11) then the filter HasEvalDiagMat for *C* is set to true and the homalgTable function DiagMat (B.1.10) will be used to set the attribute Eval.

```
InstallMethod( Eval,
        "for homalg matrices (HasEvalDiagMat)",
        [ IsHomalgMatrix and HasEvalDiagMat ],

  function( C )
    local R, RP, e, z, m, n, diag, mat;

    R := HomalgRing( C );

    RP := homalgTable( R );

    e :=  EvalDiagMat( C );

    if IsBound(RP!.DiagMat) then
        return RP!.DiagMat( e );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called DiagMat ",
               "in the homalgTable of the non-internal ring\n" );
    fi;
```

```
    #=====# can only work for homalg internal matrices #=====#

    z := Zero( R );

    m := Sum( List( e, NrRows ) );
    n := Sum( List( e, NrColumns ) );

    diag := List( [ 1 .. m ], a -> List( [ 1 .. n ], b -> z ) );

    m := 0;
    n := 0;

    for mat in e do
        diag{ [ m + 1 .. m + NrRows( mat ) ] }{ [ n + 1 .. n + NrColumns( mat ) ] }
          := Eval( mat )!.matrix;

        m := m + NrRows( mat );
        n := n + NrColumns( mat );
    od;

    return homalgInternalMatrixHull( diag );

end );
```

C.4.13 Eval (for matrices created with KroneckerMat)

▷ `Eval(C)` (method)

Returns: the `Eval` value of a homalg matrix *C*

In case the matrix was created using `KroneckerMat` (5.5.12) then the filter `HasEvalKroneckerMat` for *C* is set to true and the `homalgTable` function `KroneckerMat` (B.1.11) will be used to set the attribute `Eval`.

Code

```
InstallMethod( Eval,
        "for homalg matrices (HasEvalKroneckerMat)",
        [ IsHomalgMatrix and HasEvalKroneckerMat ],

  function( C )
    local R, RP, A, B;

    R := HomalgRing( C );

    if ( HasIsCommutative( R ) and not IsCommutative( R ) ) and
       ( HasIsSuperCommutative( R ) and not IsSuperCommutative( R ) ) then
        Info( InfoWarning, 1, "\033[01m\033[5;31;47m",
              "the Kronecker product is only defined for (super) commutative rings!",
              "\033[0m" );
    fi;

    RP := homalgTable( R );

    A :=  EvalKroneckerMat( C )[1];
```

```
    B :=  EvalKroneckerMat( C )[2];

    if IsBound(RP!.KroneckerMat) then
        return RP!.KroneckerMat( A, B );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called KroneckerMat ",
               "in the homalgTable of the non-internal ring\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#

    return homalgInternalMatrixHull(
                   KroneckerProduct( Eval( A )!.matrix, Eval( B )!.matrix ) );
    ## this was easy, thanks GAP :)

end );
```

C.4.14 Eval (for matrices created with MulMat)

▷ `Eval(C)` (method)

Returns: the `Eval` value of a homalg matrix *C*

In case the matrix was created using `\*` (5.5.13) then the filter `HasEvalMulMat` for *C* is set to true and the `homalgTable` function `MulMat` (B.1.12) will be used to set the attribute `Eval`.

Code

```
InstallMethod( Eval,
        "for homalg matrices (HasEvalMulMat)",
        [ IsHomalgMatrix and HasEvalMulMat ],

  function( C )
    local R, RP, e, a, A;

    R := HomalgRing( C );

    RP := homalgTable( R );

    e :=  EvalMulMat( C );

    a := e[1];
    A := e[2];

    if IsBound(RP!.MulMat) then
        return RP!.MulMat( a, A );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called MulMat ",
               "in the homalgTable of the non-internal ring\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#
```

```
    return a * Eval( A );

end );

InstallMethod( Eval,
        "for homalg matrices (HasEvalMulMatRight)",
        [ IsHomalgMatrix and HasEvalMulMatRight ],

  function( C )
    local R, RP, e, A, a;

    R := HomalgRing( C );

    RP := homalgTable( R );

    e :=  EvalMulMatRight( C );

    A := e[1];
    a := e[2];

    if IsBound(RP!.MulMatRight) then
        return RP!.MulMatRight( A, a );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called MulMatRight ",
               "in the homalgTable of the non-internal ring\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#

    return Eval( A ) * a;

end );
```

C.4.15 Eval (for matrices created with AddMat)

▷ `Eval(C)` (method)

Returns: the `Eval` value of a homalg matrix *C*

In case the matrix was created using `\+` (5.5.14) then the filter `HasEvalAddMat` for *C* is set to true and the `homalgTable` function `AddMat` (B.1.13) will be used to set the attribute `Eval`.

Code

```
InstallMethod( Eval,
        "for homalg matrices (HasEvalAddMat)",
        [ IsHomalgMatrix and HasEvalAddMat ],

  function( C )
    local R, RP, e, A, B;

    R := HomalgRing( C );
```

```
    RP := homalgTable( R );

    e :=  EvalAddMat( C );

    A := e[1];
    B := e[2];

    if IsBound(RP!.AddMat) then
        return RP!.AddMat( A, B );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called AddMat ",
               "in the homalgTable of the non-internal ring\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#

    return Eval( A ) + Eval( B );

end );
```

C.4.16 Eval (for matrices created with SubMat)

▷ Eval(*C*) (method)

Returns: the Eval value of a homalg matrix *C*

In case the matrix was created using \- (5.5.15) then the filter HasEvalSubMat for *C* is set to true and the homalgTable function SubMat (B.1.14) will be used to set the attribute Eval.

Code
```
InstallMethod( Eval,
        "for homalg matrices (HasEvalSubMat)",
        [ IsHomalgMatrix and HasEvalSubMat ],

  function( C )
    local R, RP, e, A, B;

    R := HomalgRing( C );

    RP := homalgTable( R );

    e :=  EvalSubMat( C );

    A := e[1];
    B := e[2];

    if IsBound(RP!.SubMat) then
        return RP!.SubMat( A, B );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called SubMat ",
               "in the homalgTable of the non-internal ring\n" );
```

```
    fi;

    #=====# can only work for homalg internal matrices #=====#

    return Eval( A ) - Eval( B );

end );
```

C.4.17 Eval (for matrices created with Compose)

▷ Eval(*C*) (method)

Returns: the Eval value of a homalg matrix *C*

In case the matrix was created using * (5.5.16) then the filter HasEvalCompose for *C* is set to true and the homalgTable function Compose (B.1.15) will be used to set the attribute Eval.

Code

```
InstallMethod( Eval,
        "for homalg matrices (HasEvalCompose)",
        [ IsHomalgMatrix and HasEvalCompose ],

  function( C )
    local R, RP, e, A, B;

    R := HomalgRing( C );

    RP := homalgTable( R );

    e :=  EvalCompose( C );

    A := e[1];
    B := e[2];

    if IsBound(RP!.Compose) then
        return RP!.Compose( A, B );
    fi;

    if not IsHomalgInternalMatrixRep( C ) then
        Error( "could not find a procedure called Compose ",
               "in the homalgTable of the non-internal ring\n" );
    fi;

    #=====# can only work for homalg internal matrices #=====#

    return Eval( A ) * Eval( B );

end );
```

Appendix D

The subpackage ResidueClassRingForHomalg as a sample ring package

D.1 The Mandatory Basic Operations

D.1.1 BasisOfRowModule (ResidueClassRing)

▷ BasisOfRowModule(*M*) (function)

Returns: a homalg matrix over the ambient ring

```
BasisOfRowModule :=
  function( M )
    local Mrel;

    Mrel := UnionOfRows( M );

    Mrel := HomalgResidueClassMatrix(
                    BasisOfRowModule( Mrel ), HomalgRing( M ) );

    return GetRidOfObsoleteRows( Mrel );

  end,
```

D.1.2 BasisOfColumnModule (ResidueClassRing)

▷ BasisOfColumnModule(*M*) (function)

Returns: a homalg matrix over the ambient ring

```
BasisOfColumnModule :=
  function( M )
    local Mrel;

    Mrel := UnionOfColumns( M );
```

```
    Mrel := HomalgResidueClassMatrix(
                    BasisOfColumnModule( Mrel ), HomalgRing( M ) );

    return GetRidOfObsoleteColumns( Mrel );

  end,
```

D.1.3 DecideZeroRows (ResidueClassRing)

▷ DecideZeroRows(*A, B*) (function)

Returns: a homalg matrix over the ambient ring

```
DecideZeroRows :=
  function( A, B )
    local Brel;

    Brel := UnionOfRows( B );

    Brel := BasisOfRowModule( Brel );

    return HomalgResidueClassMatrix(
                   DecideZeroRows( Eval( A ), Brel ), HomalgRing( A ) );

  end,
```

D.1.4 DecideZeroColumns (ResidueClassRing)

▷ DecideZeroColumns(*A, B*) (function)

Returns: a homalg matrix over the ambient ring

```
DecideZeroColumns :=
  function( A, B )
    local Brel;

    Brel := UnionOfColumns( B );

    Brel := BasisOfColumnModule( Brel );

    return HomalgResidueClassMatrix(
                   DecideZeroColumns( Eval( A ), Brel ), HomalgRing( A ) );

  end,
```

D.1.5 SyzygiesGeneratorsOfRows (ResidueClassRing)

▷ SyzygiesGeneratorsOfRows(*M*) (function)

Returns: a homalg matrix over the ambient ring

```
SyzygiesGeneratorsOfRows :=
  function( M )
```

```
    local R, ring_rel, rel, S;

    R := HomalgRing( M );

    ring_rel := RingRelations( R );

    rel := MatrixOfRelations( ring_rel );

    if IsHomalgRingRelationsAsGeneratorsOfRightIdeal( ring_rel ) then
        rel := Involution( rel );
    fi;

    rel := DiagMat( ListWithIdenticalEntries( NrColumns( M ), rel ) );

    S := SyzygiesGeneratorsOfRows( Eval( M ), rel );

    S := HomalgResidueClassMatrix( S, R );

    S := GetRidOfObsoleteRows( S );

    if IsZero( S ) then

        SetIsLeftRegular( M, true );

    fi;

    return S;

  end,
```

D.1.6 SyzygiesGeneratorsOfColumns (ResidueClassRing)

▷ SyzygiesGeneratorsOfColumns(*M*) (function)

Returns: a homalg matrix over the ambient ring

```
SyzygiesGeneratorsOfColumns :=
  function( M )
    local R, ring_rel, rel, S;

    R := HomalgRing( M );

    ring_rel := RingRelations( R );

    rel := MatrixOfRelations( ring_rel );

    if IsHomalgRingRelationsAsGeneratorsOfLeftIdeal( ring_rel ) then
        rel := Involution( rel );
    fi;

    rel := DiagMat( ListWithIdenticalEntries( NrRows( M ), rel ) );

    S := SyzygiesGeneratorsOfColumns( Eval( M ), rel );
```

```
    S := HomalgResidueClassMatrix( S, R );

    S := GetRidOfObsoleteColumns( S );

    if IsZero( S ) then

        SetIsRightRegular( M, true );

    fi;

    return S;

  end,
```

D.1.7 BasisOfRowsCoeff (ResidueClassRing)

▷ `BasisOfRowsCoeff(M, T)` (function)

Returns: a homalg matrix over the ambient ring

```
BasisOfRowsCoeff :=
  function( M, T )
    local Mrel, TT, bas, nz;

    Mrel := UnionOfRows( M );

    TT := HomalgVoidMatrix( HomalgRing( Mrel ) );

    bas := BasisOfRowsCoeff( Mrel, TT );

    bas := HomalgResidueClassMatrix( bas, HomalgRing( M ) );

    nz := NonZeroRows( bas );

    SetEval( T, CertainRows( CertainColumns( TT, [ 1 .. NrRows( M ) ] ), nz ) );

    ResetFilterObj( T, IsVoidMatrix );

    ## the generic BasisOfRowsCoeff will assume that
    ## ( NrRows( B ) = 0 ) = IsZero( B )
    return CertainRows( bas, nz );

  end,
```

D.1.8 BasisOfColumnsCoeff (ResidueClassRing)

▷ `BasisOfColumnsCoeff(M, T)` (function)

Returns: a homalg matrix over the ambient ring

```
BasisOfColumnsCoeff :=
  function( M, T )
```

```
    local Mrel, TT, bas, nz;

    Mrel := UnionOfColumns( M );

    TT := HomalgVoidMatrix( HomalgRing( Mrel ) );

    bas := BasisOfColumnsCoeff( Mrel, TT );

    bas := HomalgResidueClassMatrix( bas, HomalgRing( M ) );

    nz := NonZeroColumns( bas );

    SetEval( T, CertainColumns( CertainRows( TT, [ 1 .. NrColumns( M ) ] ), nz ) );

    ResetFilterObj( T, IsVoidMatrix );

    ## the generic BasisOfColumnsCoeff will assume that
    ## ( NrColumns( B ) = 0 ) = IsZero( B )
    return CertainColumns( bas, nz );

  end,
```

D.1.9 DecideZeroRowsEffectively (ResidueClassRing)

▷ DecideZeroRowsEffectively(*A, B, T*) (function)

Returns: a homalg matrix over the ambient ring

Code

```
DecideZeroRowsEffectively :=
  function( A, B, T )
    local Brel, TT, red;

    Brel := UnionOfRows( B );

    TT := HomalgVoidMatrix( HomalgRing( Brel ) );

    red := DecideZeroRowsEffectively( Eval( A ), Brel, TT );

    SetEval( T, CertainColumns( TT, [ 1 .. NrRows( B ) ] ) );

    ResetFilterObj( T, IsVoidMatrix );

    return HomalgResidueClassMatrix( red, HomalgRing( A ) );

  end,
```

D.1.10 DecideZeroColumnsEffectively (ResidueClassRing)

▷ DecideZeroColumnsEffectively(*A, B, T*) (function)

Returns: a homalg matrix over the ambient ring

Code

```
DecideZeroColumnsEffectively :=
  function( A, B, T )
    local Brel, TT, red;

    Brel := UnionOfColumns( B );

    TT := HomalgVoidMatrix( HomalgRing( Brel ) );

    red := DecideZeroColumnsEffectively( Eval( A ), Brel, TT );

    SetEval( T, CertainRows( TT, [ 1 .. NrColumns( B ) ] ) );

    ResetFilterObj( T, IsVoidMatrix );

    return HomalgResidueClassMatrix( red, HomalgRing( A ) );

  end,
```

D.1.11 RelativeSyzygiesGeneratorsOfRows (ResidueClassRing)

▷ `RelativeSyzygiesGeneratorsOfRows(M, M2)` (function)

Returns: a homalg matrix over the ambient ring

Code

```
RelativeSyzygiesGeneratorsOfRows :=
  function( M, M2 )
    local M2rel, S;

    M2rel := UnionOfRows( M2 );

    S := SyzygiesGeneratorsOfRows( Eval( M ), M2rel );

    S := HomalgResidueClassMatrix( S, HomalgRing( M ) );

    S := GetRidOfObsoleteRows( S );

    if IsZero( S ) then

        SetIsLeftRegular( M, true );

    fi;

    return S;

  end,
```

D.1.12 RelativeSyzygiesGeneratorsOfColumns (ResidueClassRing)

▷ `RelativeSyzygiesGeneratorsOfColumns(M, M2)` (function)

Returns: a homalg matrix over the ambient ring

```
RelativeSyzygiesGeneratorsOfColumns :=
  function( M, M2 )
    local M2rel, S;

    M2rel := UnionOfColumns( M2 );

    S := SyzygiesGeneratorsOfColumns( Eval( M ), M2rel );

    S := HomalgResidueClassMatrix( S, HomalgRing( M ) );

    S := GetRidOfObsoleteColumns( S );

    if IsZero( S ) then

        SetIsRightRegular( M, true );

    fi;

    return S;

  end,
```

D.2 The Mandatory Tool Operations

Here we list those matrix operations for which homalg provides no fallback method.

D.2.1 InitialMatrix (ResidueClassRing)

▷ InitialMatrix() (function)

Returns: a homalg matrix over the ambient ring
(→ InitialMatrix (B.1.1))

```
InitialMatrix := C -> HomalgInitialMatrix(
                      NrRows( C ), NrColumns( C ), AmbientRing( HomalgRing( C ) ) ),
```

D.2.2 InitialIdentityMatrix (ResidueClassRing)

▷ InitialIdentityMatrix() (function)

Returns: a homalg matrix over the ambient ring
(→ InitialIdentityMatrix (B.1.2))

```
InitialIdentityMatrix := C -> HomalgInitialIdentityMatrix(
        NrRows( C ), AmbientRing( HomalgRing( C ) ) ),
```

D.2.3 ZeroMatrix (ResidueClassRing)

▷ ZeroMatrix() (function)

Returns: a homalg matrix over the ambient ring

(→ ZeroMatrix (B.1.3))

Code
```
ZeroMatrix := C -> HomalgZeroMatrix(
                NrRows( C ), NrColumns( C ), AmbientRing( HomalgRing( C ) ) ),
```

D.2.4 IdentityMatrix (ResidueClassRing)

▷ IdentityMatrix() (function)

Returns: a homalg matrix over the ambient ring

(→ IdentityMatrix (B.1.4))

Code
```
IdentityMatrix := C -> HomalgIdentityMatrix(
        NrRows( C ), AmbientRing( HomalgRing( C ) ) ),
```

D.2.5 Involution (ResidueClassRing)

▷ Involution() (function)

Returns: a homalg matrix over the ambient ring

(→ Involution (B.1.5))

Code
```
Involution :=
  function( M )
    local N, R;

    N := Involution( Eval( M ) );

    R := HomalgRing( N );

    if not ( HasIsCommutative( R ) and IsCommutative( R ) and
             HasIsReducedModuloRingRelations( M ) and
             IsReducedModuloRingRelations( M ) ) then

        ## reduce the matrix N w.r.t. the ring relations
        N := DecideZero( N, HomalgRing( M ) );
    fi;

    return N;

  end,
```

D.2.6 CertainRows (ResidueClassRing)

▷ CertainRows() (function)

Returns: a homalg matrix over the ambient ring

(→ CertainRows (B.1.6))

Code
```
CertainRows :=
  function( M, plist )
    local N;
```

```
    N := CertainRows( Eval( M ), plist );

    if not ( HasIsReducedModuloRingRelations( M ) and
             IsReducedModuloRingRelations( M ) ) then

        ## reduce the matrix N w.r.t. the ring relations
        N := DecideZero( N, HomalgRing( M ) );
    fi;

    return N;

  end,
```

D.2.7 CertainColumns (ResidueClassRing)

▷ `CertainColumns()` (function)

Returns: a homalg matrix over the ambient ring

(→ `CertainColumns` (B.1.7))

```
CertainColumns :=
  function( M, plist )
    local N;

    N := CertainColumns( Eval( M ), plist );

    if not ( HasIsReducedModuloRingRelations( M ) and
             IsReducedModuloRingRelations( M ) ) then

        ## reduce the matrix N w.r.t. the ring relations
        N := DecideZero( N, HomalgRing( M ) );
    fi;

    return N;

  end,
```

D.2.8 UnionOfRows (ResidueClassRing)

▷ `UnionOfRows()` (function)

Returns: a homalg matrix over the ambient ring

(→ `UnionOfRows` (B.1.8))

```
UnionOfRows :=
  function( A, B )
    local N;

    N := UnionOfRows( Eval( A ), Eval( B ) );

    if not ForAll( [ A, B ], HasIsReducedModuloRingRelations and
               IsReducedModuloRingRelations ) then
```

```
        ## reduce the matrix N w.r.t. the ring relations
        N := DecideZero( N, HomalgRing( A ) );
    fi;

    return N;

  end,
```

D.2.9 UnionOfColumns (ResidueClassRing)

▷ UnionOfColumns() (function)

Returns: a homalg matrix over the ambient ring

(→ UnionOfColumns (B.1.9))

Code

```
UnionOfColumns :=
  function( A, B )
    local N;

    N := UnionOfColumns( Eval( A ), Eval( B ) );

    if not ForAll( [ A, B ], HasIsReducedModuloRingRelations and
                   IsReducedModuloRingRelations ) then

        ## reduce the matrix N w.r.t. the ring relations
        N := DecideZero( N, HomalgRing( A ) );
    fi;

    return N;

  end,
```

D.2.10 DiagMat (ResidueClassRing)

▷ DiagMat() (function)

Returns: a homalg matrix over the ambient ring

(→ DiagMat (B.1.10))

Code

```
DiagMat :=
  function( e )
    local N;

    N := DiagMat( List( e, Eval ) );

    if not ForAll( e, HasIsReducedModuloRingRelations and
                   IsReducedModuloRingRelations ) then

        ## reduce the matrix N w.r.t. the ring relations
        N := DecideZero( N, HomalgRing( e[1] ) );
    fi;

    return N;
```

```
  end,
```

D.2.11 KroneckerMat (ResidueClassRing)

▷ KroneckerMat() (function)

Returns: a homalg matrix over the ambient ring

(→ KroneckerMat (B.1.11))

```
KroneckerMat :=
  function( A, B )
    local N;

    N := KroneckerMat( Eval( A ), Eval( B ) );

    if not ForAll( [ A, B ], HasIsReducedModuloRingRelations and
               IsReducedModuloRingRelations ) then

        ## reduce the matrix N w.r.t. the ring relations
        N := DecideZero( N, HomalgRing( A ) );
    fi;

    return N;

  end,
```

D.2.12 MulMat (ResidueClassRing)

▷ MulMat() (function)

Returns: a homalg matrix over the ambient ring

(→ MulMat (B.1.12))

```
MulMat :=
  function( a, A )

    return DecideZero( EvalRingElement( a ) * Eval( A ), HomalgRing( A ) );

  end,
MulMatRight :=
  function( A, a )

    return DecideZero( Eval( A ) * EvalRingElement( a ), HomalgRing( A ) );

  end,
```

D.2.13 AddMat (ResidueClassRing)

▷ AddMat() (function)

Returns: a homalg matrix over the ambient ring

(→ AddMat (B.1.13))

```
AddMat :=
  function( A, B )

    return DecideZero( Eval( A ) + Eval( B ), HomalgRing( A ) );

  end,
```

D.2.14 SubMat (ResidueClassRing)

▷ SubMat() (function)

Returns: a homalg matrix over the ambient ring

(→ SubMat (B.1.14))

```
SubMat :=
  function( A, B )

    return DecideZero( Eval( A ) - Eval( B ), HomalgRing( A ) );

  end,
```

D.2.15 Compose (ResidueClassRing)

▷ Compose() (function)

Returns: a homalg matrix over the ambient ring

(→ Compose (B.1.15))

```
Compose :=
  function( A, B )

    return DecideZero( Eval( A ) * Eval( B ), HomalgRing( A ) );

  end,
```

D.2.16 IsZeroMatrix (ResidueClassRing)

▷ IsZeroMatrix(*M*) (function)

Returns: true or false

(→ IsZeroMatrix (B.1.16))

```
IsZeroMatrix := M -> IsZero( DecideZero( Eval( M ), HomalgRing( M ) ) ),
```

D.2.17 NrRows (ResidueClassRing)

▷ NrRows(*C*) (function)

Returns: a nonnegative integer

(→ NrRows (B.1.17))

```
NrRows := C -> NrRows( Eval( C ) ),
```

D.2.18 NrColumns (ResidueClassRing)

▷ `NrColumns(C)` (function)

Returns: a nonnegative integer

(→ `NrColumns` (B.1.18))

Code
```
NrColumns := C -> NrColumns( Eval( C ) ),
```

D.2.19 Determinant (ResidueClassRing)

▷ `Determinant(C)` (function)

Returns: an element of ambient homalg ring

(→ `Determinant` (B.1.19))

Code
```
Determinant := C -> DecideZero( Determinant( Eval( C ) ), HomalgRing( C ) ),
```

D.3 Some of the Recommended Tool Operations

Here we list those matrix operations for which homalg does provide a fallback method. But specifying the below `homalgTable` functions increases the performance by replacing the fallback method.

D.3.1 AreEqualMatrices (ResidueClassRing)

▷ `AreEqualMatrices(A, B)` (function)

Returns: `true` or `false`

(→ `AreEqualMatrices` (B.2.1))

Code
```
AreEqualMatrices :=
  function( A, B )

    return IsZero( DecideZero( Eval( A ) - Eval( B ), HomalgRing( A ) ) );

  end,
```

D.3.2 IsOne (ResidueClassRing)

▷ `IsOne(M)` (function)

Returns: `true` or `false`

(→ `IsIdentityMatrix` (B.2.2))

Code
```
IsIdentityMatrix := M ->
          IsOne( DecideZero( Eval( M ), HomalgRing( M ) ) ),
```

D.3.3 IsDiagonalMatrix (ResidueClassRing)

▷ `IsDiagonalMatrix(M)` (function)

Returns: `true` or `false`

(→ `IsDiagonalMatrix` (B.2.3))

```
IsDiagonalMatrix := M ->
          IsDiagonalMatrix( DecideZero( Eval( M ), HomalgRing( M ) ) ),
```

D.3.4 ZeroRows (ResidueClassRing)

▷ ZeroRows(*C*) (function)

Returns: a homalg matrix over the ambient ring

(→ ZeroRows (B.2.4))

```
ZeroRows := C -> ZeroRows( DecideZero( Eval( C ), HomalgRing( C ) ) ),
```

D.3.5 ZeroColumns (ResidueClassRing)

▷ ZeroColumns(*C*) (function)

Returns: a homalg matrix over the ambient ring

(→ ZeroColumns (B.2.5))

```
ZeroColumns := C -> ZeroColumns( DecideZero( Eval( C ), HomalgRing( C ) ) ),
```

Appendix E

Debugging MatricesForHomalg

Beside the GAP builtin debugging facilities (→ (**Reference: Debugging and Profiling Facilities**)) MatricesForHomalg provides two ways to debug the computations.

E.1 Increase the assertion level

MatricesForHomalg comes with numerous builtin assertion checks. They are activated if the user increases the assertion level using

`SetAssertionLevel(` *level* `);`

(→ (**Reference: SetAssertionLevel**)), where *level* is one of the values below:

level	description
0	no assertion checks whatsoever
4	assertions about basic matrix operations are checked (→ Appendix A) (these are among the operations often delegated to external systems)

In particular, if MatricesForHomalg delegates matrix operations to an external system then `SetAssertionLevel(` 4 `);` can be used to let MatricesForHomalg debug the external system.

Below you can find the record of the available level-4 assertions, which is a GAP-component of every homalg ring. Each assertion can thus be overwritten by package developers or even ordinary users.

Code

```
asserts :=
  rec(
      BasisOfRowModule :=
        function( B ) return ( NrRows( B ) = 0 ) = IsZero( B ); end,

      BasisOfColumnModule :=
        function( B ) return ( NrColumns( B ) = 0 ) = IsZero( B ); end,
```

```
BasisOfRowsCoeff :=
  function( B, T, M ) return B = T * M; end,

BasisOfColumnsCoeff :=
  function( B, M, T ) return B = M * T; end,

DecideZeroRows_Effectively :=
  function( M, A, B ) return M = DecideZeroRows( A, B ); end,

DecideZeroColumns_Effectively :=
  function( M, A, B ) return M = DecideZeroColumns( A, B ); end,

DecideZeroRowsEffectively :=
  function( M, A, T, B ) return M = A + T * B; end,

DecideZeroColumnsEffectively :=
  function( M, A, B, T ) return M = A + B * T; end,

DecideZeroRowsWRTNonBasis :=
  function( B )
    local R;
    R := HomalgRing( B );
    if not ( HasIsBasisOfRowsMatrix( B ) and
             IsBasisOfRowsMatrix( B ) ) and
       IsBound( R!.DecideZeroWRTNonBasis ) then
        if R!.DecideZeroWRTNonBasis = "warn" then
            Info( InfoWarning, 1,
                  "about to reduce with respect to a matrix",
                  "with IsBasisOfRowsMatrix not set to true" );
        elif R!.DecideZeroWRTNonBasis = "error" then
            Error( "about to reduce with respect to a matrix",
                   "with IsBasisOfRowsMatrix not set to true\n" );
        fi;
    fi;
  end,

DecideZeroColumnsWRTNonBasis :=
  function( B )
    local R;
    R := HomalgRing( B );
    if not ( HasIsBasisOfColumnsMatrix( B ) and
             IsBasisOfColumnsMatrix( B ) ) and
       IsBound( R!.DecideZeroWRTNonBasis ) then
        if R!.DecideZeroWRTNonBasis = "warn" then
            Info( InfoWarning, 1,
                  "about to reduce with respect to a matrix",
                  "with IsBasisOfColumnsMatrix not set to true" );
        elif R!.DecideZeroWRTNonBasis = "error" then
            Error( "about to reduce with respect to a matrix",
                   "with IsBasisOfColumnsMatrix not set to true\n" );
        fi;
    fi;
  end,
```

```
    ReducedBasisOfRowModule :=
      function( M, B )
        return GenerateSameRowModule( B, BasisOfRowModule( M ) );
      end,

    ReducedBasisOfColumnModule :=
      function( M, B )
        return GenerateSameColumnModule( B, BasisOfColumnModule( M ) );
      end,

    ReducedSyzygiesGeneratorsOfRows :=
      function( M, S )
        return GenerateSameRowModule( S, SyzygiesGeneratorsOfRows( M ) );
      end,

    ReducedSyzygiesGeneratorsOfColumns :=
      function( M, S )
        return GenerateSameColumnModule( S, SyzygiesGeneratorsOfColumns( M ) );
      end,

  );
```

E.2 Using homalgMode

E.2.1 homalgMode

▷ `homalgMode(`*`str[, str2]`*`)` (method)

This function sets different modes which influence how much of the basic matrix operations and the logical matrix methods become visible (→ Appendices A, C). Handling the string *str* is *not* case-sensitive. If a second string *str2* is given, then `homalgMode(` *str2* `)` is invoked at the end. In case you let homalg delegate matrix operations to an external system the you might also want to check `homalgIOMode` in the HomalgToCAS package manual.

str	*str* (long form)	mode description
""	""	the default mode, i.e. the computation protocol won't be visible (`homalgMode(`) is a short form for `homalgMode(` ""))
"b"	"basic"	make the basic matrix operations visible + `homalgMode(` "logic")
"d"	"debug"	same as "basic" but also makes `Row/ColumnReducedEchelonForm` visible
"l"	"logic"	make the logical methods in LIMAT and COLEM visible

All modes other than the "default"-mode only set their specific values and leave the other values untouched, which allows combining them to some extent. This also means that in order to get from one mode to a new mode (without the aim to combine them) one needs to reset to the "default"-mode first. This can be done using homalgMode("", new_mode);

Code

```
InstallGlobalFunction( homalgMode,
  function( arg )
    local nargs, mode, s;

    nargs := Length( arg );

    if nargs = 0 or ( IsString( arg[1] ) and arg[1] = "" ) then
        mode := "default";
    elif IsString( arg[1] ) then        ## now we know, the string is not empty
        s := arg[1];
        if LowercaseString( s{[1]} ) = "b" then
            mode := "basic";
        elif LowercaseString( s{[1]} ) = "d" then
            mode := "debug";
        elif LowercaseString( s{[1]} ) = "l" then
            mode := "logic";
        else
            mode := "";
        fi;
    else
        Error( "the first argument must be a string\n" );
    fi;

    if mode = "default" then
        HOMALG_MATRICES.color_display := false;
        for s in HOMALG_MATRICES.matrix_logic_infolevels do
            SetInfoLevel( s, 1 );
        od;
        SetInfoLevel( InfoHomalgBasicOperations, 1 );
    elif mode = "basic" then
        SetInfoLevel( InfoHomalgBasicOperations, 3 );
        homalgMode( "logic" );
    elif mode = "debug" then
        SetInfoLevel( InfoHomalgBasicOperations, 4 );
        homalgMode( "logic" );
    elif mode = "logic" then
        HOMALG_MATRICES.color_display := true;
        for s in HOMALG_MATRICES.matrix_logic_infolevels do
            SetInfoLevel( s, 2 );
        od;
    fi;

    if nargs > 1 and IsString( arg[2] ) then
        homalgMode( arg[2] );
    fi;

end );
```

Appendix F

Overview of the MatricesForHomalg Package Source Code

F.1 Rings, Ring Maps, Matrices, Ring Relations

Filename `.gd`/`.gi`	Content
`homalg`	definitions of the basic GAP4 categories and some tool functions (e.g. `homalgMode`)
`homalgTable`	dictionaries between MatricesForHomalg and the computing engines
`HomalgRing`	internal and external rings
`HomalgRingMap`	ring maps
`HomalgMatrix`	internal and external matrices
`HomalgRingRelations`	a set of ring relations

Table: *The MatricesForHomalg package files*

F.2 The Low Level Algorithms

In the following CAS or CASystem mean computer algebra systems.

Filename .gd/.gi	Content
Tools	the elementary matrix operations that can be overwritten using the homalgTable (and hence delegable even to other CASystems)
Service	the three operations: basis, reduction, and syzygies; they can also be overwritten using the homalgTable (and hence delegable even to other CASystems)
Basic	higher level operations for matrices (cannot be overwritten using the homalgTable)

Table: *The MatricesForHomalg package files (continued)*

F.3 Logical Implications for MatricesForHomalg Objects

Filename .gd/.gi	Content
LIRNG	logical implications for rings
LIMAP	logical implications for ring maps
LIMAT	logical implications for matrices
COLEM	clever operations for lazy evaluated matrices

Table: *The MatricesForHomalg package files (continued)*

F.4 The subpackage ResidueClassRingForHomalg

Filename .gd/.gi	Content
ResidueClassRingForHomalg	some global variables
ResidueClassRing	residue class rings, their elements, and matrices, together with their constructors and operations
ResidueClassRingTools	the elementary matrix operations for matrices over residue class rings
ResidueClassRingBasic	the three operations: basis, reduction, and syzygies for matrices over residue class rings

Table: *The MatricesForHomalg package files (continued)*

F.5 The homalgTable for GAP4 built-in rings

For the purposes of homalg, the ring of integers is, at least up till now, the only ring which is properly supported in GAP4. The GAP4 built-in cababilities for polynomial rings (also univariate) and group rings do not statisfy the minimum requirements of homalg. The GAP4 package Gauss enables GAP to fullfil the homalg requirements for prime fields, and $\mathbb{Z}/p^n$.

Filename .gi	Content
`Integers`	the homalgTable for the ring of integers

Table: *The MatricesForHomalg package files (continued)*

References

[BR08] Mohamed Barakat and Daniel Robertz. homalg – A Meta-Package for Homological Algebra. *J. Algebra Appl.*, 7(3):299–317, 2008. arXiv:math.AC/0701146. 49, 50

[MR01] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings*, volume 30 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, revised edition, 2001. With the cooperation of L. W. Small. 22

Index