Congruence

Congruence subgroups of $SL_2(\mathbb{Z})$

Version 1.2.2

6 February 2018

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Abstract

The GAP package Congruence provides functionality to work with congruence subgroups of $SL_2(\mathbb{Z})$.

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If you obtained Congruence, we would be grateful for a short notification sent to one of the authors.

If you publish a result which was partially obtained with the usage of Congruence, please cite it in the following form:


Acknowledgements

We are very grateful to Mong-Lung Lang, Chong-Hai Lim and Ser Peow Tan for their comments provided while implementing algorithms from [LLT95a] and [LLT95b], and to Francqui Stichting (Belgium) for the support of the third author.
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Chapter 1

Introduction

1.1 General aims of Congruence package

The GAP package Congruence provides functions to construct several types of canonical congruence subgroups in $\text{SL}_2(\mathbb{Z})$, and also intersections of a finite number of such subgroups.

Furthermore, it implements the algorithm for generating Farey symbols for congruence subgroups and using them to produce a system of independent generators for these subgroups.

Using the package, one can also determine indices of congruence subgroups and their intersections in $\text{SL}_2(\mathbb{Z})$ and in other congruence subgroups, generate their random elements and check element memberships. Success of other group theoretical constructions mostly depends on whether they could be expressed in terms of group generators or not.

For the theoretical background, we refer to [LLT95b], [LLT95a], [CLLT93] and [Kul91].

1.2 Installation and system requirements

Congruence is distributed in standard formats (tar.gz, tar.bz2, -win.zip) and can be obtained from http://www.cs.st-andrews.ac.uk/~alexk/congruence/.

Congruence does not use external binaries and, therefore, works without restrictions on the operating system. It requires at least version GAP 4.5, and no compatibility with previous releases of GAP 4 is guaranteed.

Installation of the package is standard and follows the guidelines from the GAP manual (see (Reference: Installing a GAP Package). After the package is installed, you can start GAP and load the Congruence package using the command:

```gap
gap> LoadPackage("congruence");
```
Chapter 2

Construction of congruence subgroups

The package Congruence provides functions to construct several types of canonical congruence subgroups in $SL_2(\mathbb{Z})$, and also intersections of a finite number of such subgroups. They will return a matrix group in the category IsCongruenceSubgroup, which is defined as a subcategory of IsMatrixGroup, and which will have a distinguishing property determining whether it is a congruence subgroup of one of the canonical types, or an intersection of such congruence subgroups (if it can not be reduced to one of the canonical congruence subgroups). To start to work with the package, you need first to load it as follows:

```gap
    gap> LoadPackage("congruence");
    true
```

2.1 Construction of congruence subgroups

2.1.1 PrincipalCongruenceSubgroup

\[ \text{PrincipalCongruenceSubgroup}(N) \]

Returns the principal congruence subgroup $\Gamma(N)$ of level $N$ in $SL_2(\mathbb{Z})$. This subgroup consists of all matrices of the form

\[
\begin{pmatrix}
1 + Na & Nb \\
Nc & 1 + Nd
\end{pmatrix}
\]

where $a,b,c,d$ are integers. The returned group will have the property IsPrincipalCongruenceSubgroup (2.2.1).
2.1.2 CongruenceSubgroupGamma0

\[ \text{CongruenceSubgroupGamma0}(N) \]

Returns the congruence subgroup \( \Gamma_0(N) \) of level \( N \) in \( SL_2(\mathbb{Z}) \).

This subgroup consists of all matrices of the form

\[
\begin{pmatrix}
a & b \\
Nc & d
\end{pmatrix}
\]

where \( a, b, c, d \) are integers. The returned group will have the property \text{IsCongruenceSubgroupGamma0 (2.2.2)}.

Example

```gap
gap> G0_4:=CongruenceSubgroupGamma0(4);
<congruence subgroup CongruenceSubgroupGamma_0(4) in SL_2(Z)>
```

2.1.3 CongruenceSubgroupGammaUpper0

\[ \text{CongruenceSubgroupGammaUpper0}(N) \]

Returns the congruence subgroup \( \Gamma^0(N) \) of level \( N \) in \( SL_2(\mathbb{Z}) \).

This subgroup consists of all matrices of the form

\[
\begin{pmatrix}
a & Nb \\
c & d
\end{pmatrix}
\]
where \( a, b, c, d \) are integers. The returned group will have the property \( \text{IsCongruenceSubgroupGammaUpper0} (2.2.3) \).

Example

\begin{verbatim}
gap> GU0_2:=CongruenceSubgroupGammaUpper0(2);
<congruence subgroup CongruenceSubgroupGamma^0(2) in SL_2(Z)>
\end{verbatim}

\subsection{2.1.4 \texttt{CongruenceSubgroupGamma1}}

\texttt{CongruenceSubgroupGamma1(\( N \))}

Returns the congruence subgroup \( \Gamma_1(\( N \)) \) of level \( N \) in \( SL_2(\mathbb{Z}) \).

This subgroup consists of all matrices of the form

\[
\begin{pmatrix}
1 + Na & b \\
Nc & 1 + Nd
\end{pmatrix}
\]

where \( a, b, c, d \) are integers. The returned group will have the property \( \text{IsCongruenceSubgroupGamma1} (2.2.4) \).

Example

\begin{verbatim}
gap> G1_6:=CongruenceSubgroupGamma1(6);
<congruence subgroup CongruenceSubgroupGamma_1(6) in SL_2(Z)>
\end{verbatim}

\subsection{2.1.5 \texttt{CongruenceSubgroupGammaUpper1}}

\texttt{CongruenceSubgroupGammaUpper1(\( N \))}

Returns the congruence subgroup \( \Gamma^1(\( N \)) \) of level \( N \) in \( SL_2(\mathbb{Z}) \).

This subgroup consists of all matrices of the form

\[
\begin{pmatrix}
1 + Na & Nb \\
c & 1 + Nd
\end{pmatrix}
\]

where \( a, b, c, d \) are integers. The returned group will have the property \( \text{IsCongruenceSubgroupGammaUpper1} (2.2.5) \).

Example

\begin{verbatim}
gap> GU1_4:=CongruenceSubgroupGammaUpper1(4);
<congruence subgroup CongruenceSubgroupGamma^1(4) in SL_2(Z)>
\end{verbatim}
2.1.6 IntersectionOfCongruenceSubgroups

\[ \text{IntersectionOfCongruenceSubgroups}(G_1, G_2, \ldots, G_N) \]

The returned group will have the property IsIntersectionOfCongruenceSubgroups (2.2.6).

Example

\[ \text{gap} > I := \text{IntersectionOfCongruenceSubgroups}(G_0_4, G_U1_4); \]
\[ \text{<principal congruence subgroup of level 4 in SL}_2(\mathbb{Z})> \]
\[ \text{gap} > J := \text{IntersectionOfCongruenceSubgroups}(G_0_4, G_1_6); \]
\[ \text{<intersection of congruence subgroups of resulting level 12 in SL}_2(\mathbb{Z})> \]

2.2 Properties of congruence subgroups

A congruence subgroup constructed by one of the five above listed functions will have certain properties determining its type. These properties will be used for method selection by Congruence algorithms. Note that they do not provide an actual test whether a certain matrix group is a congruence subgroup or not.

2.2.1 IsPrincipalCongruenceSubgroup

\[ \text{IsPrincipalCongruenceSubgroup}(G) \]

Example

\[ \text{gap} > \text{IsPrincipalCongruenceSubgroup}(G_8); \]
\[ \text{true} \]
\[ \text{gap} > \text{IsPrincipalCongruenceSubgroup}(G_0_4); \]
\[ \text{false} \]
\[ \text{gap} > \text{IsPrincipalCongruenceSubgroup}(I); \]
\[ \text{true} \]

2.2.2 IsCongruenceSubgroupGamma0

\[ \text{IsCongruenceSubgroupGamma0}(G) \]
For a congruence subgroup $G$ in the category IsCongruenceSubgroup, returns true if $G$ was constructed by CongruenceSubgroupGamma0 (2.1.2) (or reduced to one as a result of an intersection) and returns false otherwise.

### 2.2.3 IsCongruenceSubgroupGammaUpper0

$\triangleright$ IsCongruenceSubgroupGammaUpper0($G$) (property)

For a congruence subgroup $G$ in the category IsCongruenceSubgroup, returns true if $G$ was constructed by CongruenceSubgroupGammaUpper0 (2.1.3) (or reduced to one as a result of an intersection) and returns false otherwise.

### 2.2.4 IsCongruenceSubgroupGamma1

$\triangleright$ IsCongruenceSubgroupGamma1($G$) (property)

For a congruence subgroup $G$ in the category IsCongruenceSubgroup, returns true if $G$ was constructed by CongruenceSubgroupGamma1 (2.1.4) (or reduced to one as a result of an intersection) and returns false otherwise.

### 2.2.5 IsCongruenceSubgroupGammaUpper1

$\triangleright$ IsCongruenceSubgroupGammaUpper1($G$) (property)

For a congruence subgroup $G$ in the category IsCongruenceSubgroup, returns true if $G$ was constructed by CongruenceSubgroupGammaUpper1 (2.1.5) (or reduced to one as a result of an intersection) and returns false otherwise.

### 2.2.6 IsIntersectionOfCongruenceSubgroups

$\triangleright$ IsIntersectionOfCongruenceSubgroups($G$) (property)

For a congruence subgroup $G$ in the category IsCongruenceSubgroup, returns true if $G$ was constructed by IntersectionOfCongruenceSubgroups (2.1.6) and without being one of the canonical congruence subgroups, otherwise it returns false.

Example  

```gap
gap> IsIntersectionOfCongruenceSubgroups(I);  
false  
gap> IsIntersectionOfCongruenceSubgroups(J);  
true  
```

### 2.3 Attributes of congruence subgroups

The next three attributes store key properties of congruence subgroups.
2.3.1 LevelOfCongruenceSubgroup

\[ \text{LevelOfCongruenceSubgroup}(G) \]

Stores the level of the congruence subgroup \( G \). The (arithmetic) level of a congruence subgroup \( G \) is the smallest positive number \( N \) such that \( G \) contains the principal congruence subgroup of level \( N \).

Example

```gap
gap> LevelOfCongruenceSubgroup(G_8); 8
gap> LevelOfCongruenceSubgroup(G1_6); 6
gap> LevelOfCongruenceSubgroup(I); 4
gap> LevelOfCongruenceSubgroup(J); 12
```

2.3.2 IndexInSL2Z

\[ \text{IndexInSL2Z}(G) \]

Stores the index of the congruence subgroup \( G \) in \( SL_2(\mathbb{Z}) \).

Example

```gap
gap> IndexInSL2Z(G_8); 384
gap> G_2:=PrincipalCongruenceSubgroup(2);
<principal congruence subgroup of level 2 in SL_2(\mathbb{Z})>
gap> IndexInSL2Z(G_2); 12
gap> IndexInSL2Z(GU1_4); 12
```

2.3.3 DefiningCongruenceSubgroups

\[ \text{DefiningCongruenceSubgroups}(G) \]

Returns: list of congruence subgroups

For an intersection of congruence subgroups, returns the list of congruence subgroups forming this intersection. For a canonical congruence subgroup returns a list of length one containing that subgroup.

Example

```gap
gap> DefiningCongruenceSubgroups(J);
[ <congruence subgroup CongruenceSubgroupGamma_0(4) in SL_2(\mathbb{Z})>,
  <congruence subgroup CongruenceSubgroupGamma_1(6) in SL_2(\mathbb{Z})> ]
gap> P:=PrincipalCongruenceSubgroup(6);
<principal congruence subgroup of level 6 in SL_2(\mathbb{Z})>
```
2.4 Operations for congruence subgroups

Congruence installs several special methods for operations already available in GAP.

2.4.1 Random (one and two argument versions)

\( \text{Random}(G) \)  
\( \text{Random}(G, m) \)

For a congruence subgroup \( G \) in the category \( \text{IsCongruenceSubgroup} \), returns random element. In the two-argument form, the second parameter will control the absolute value of randomly selected entries of the matrix.

\[
gap> \text{Random}(G_2) \in G_2; \\
true \\
gap> \text{Random}(G_8,2) \in G_8; \\
true
\]

2.4.2 \( \in \)

\( \text{\( \in(m, G) \)} \)

It is easy to implement the membership test for congruence subgroups and their intersections.

\[
gap> \in([ [21, 10], [2, 1] ], G_2); \\
true \\
gap> \in([ [21, 10], [2, 1] ], G_8); \\
false
\]

2.4.3 CanEasilyCompareCongruenceSubgroups

\( \text{CanEasilyCompareCongruenceSubgroups}(G, H) \)

For congruence subgroups \( G,H \) in the category \( \text{IsCongruenceSubgroup} \), returns true if \( G \) and \( H \) are of the same type listed in \( \text{PrincipalCongruenceSubgroup} \) \((2.1.1) \) \( \Rightarrow \) \( \text{CongruenceSubgroupGammaUpper1} \) \((2.1.5) \) and have the same \( \text{LevelOfCongruenceSubgroup} \)
(2.3.1) or if $G$ and $H$ are of the type IntersectionOfCongruenceSubgroups (2.1.6) and the groups from DefiningCongruenceSubgroups (2.3.3) are in one to one correspondence, otherwise it returns false.

Example

```gap
gap> CanEasilyCompareCongruenceSubgroups(G_8, I); false
```

### 2.4.4 IsSubset

**IsSubset** $(G, H)$

Congruence provides methods for *IsSubset* for congruence subgroups. *IsSubset* returns true if $H$ is a subset of $G$. These methods make it possible to use *IsSubgroup* operation for congruence subgroups.

Example

```gap
gap> IsSubset(G_2, G_8); true
gap> IsSubset(G_8, G_2); false
```

### 2.4.5 Index

**Index** $(G, H)$

If a congruence subgroup $H$ is a subgroup of a congruence subgroup $G$, we can easily compute the index of $H$ in $G$, since we know the index of both subgroups in $SL_2(\mathbb{Z})$.

Example

```gap
gap> Index(G_2, G_8); 32
```
Chapter 3

Farey symbols and their properties

A Farey symbol is a compact and useful way to represent a subgroup of finite index in $SL_2(\mathbb{Z})$ from which one can deduce independent generators for this subgroup. It consists of two components, namely a so-called generalised Farey sequence (gfs) and an ordered list of labels, giving additional structure to the gfs.

A generalised Farey sequence (g.F.S.) is an ordered list of the form $-\infty, x_0, x_1, \ldots, x_n, \infty$, where

1. the $x_i = a_i/b_i$ are rational numbers in reduced form arranged in increasing order for $i = 0, \ldots, n$;
2. $x_0, \ldots, x_n \in \mathbb{Z}$, and some $x_i = 0$;
3. we define $x_{-1} = -\infty = -1/0$ and $x_{n+1} = \infty = 1/0$;
4. $a_{i+1}b_i - a_i b_{i+1} = 1$ for $i = -1, \ldots, n$.

The ordered list of labels of a Farey symbol gives an additional structure to the gfs. The labels correspond to each consecutive pair of $x_i$’s and are of the following types:

1. even,
2. odd,
3. a natural number, which occurs in the list of labels exactly twice or not at all.

Note that the actual values of numerical labels are not important; it is the pairing of two intervals that matters.

The package `Congruence` provides functions to construct Farey symbols by the given generalised Farey sequence and corresponding list of labels. The returned Farey symbol will belong to the category `IsFareySymbol` and will have the representation `IsFareySymbolDefaultRep`.

3.1 Construction of Farey symbols

3.1.1 FareySymbolByData

```
> FareySymbolByData(gfs, labels)
```

This constructor creates the Farey symbol with the given generalized Farey sequence and list of labels. It also checks conditions from the definition of Farey symbol and returns an error if they are not satisfied. The data used to create the Farey symbol are stored as its attributes `GeneralizedFareySequence` (3.2.1) and `LabelsOfFareySymbol` (3.2.4).

```
gap> fs:=FareySymbolByData([infinity,0,1,2,infinity],[1,2,2,1]);
```

Example
3.1.2 IsValidFareySymbol

\[ \text{IsValidFareySymbol}(fs) \]

This function is used in FareySymbolByData (3.1.1) to validate its output.

Example

\[ \text{gap> IsValidFareySymbol}(fs); \]
\[ \text{true} \]

3.2 Properties of Farey symbols

3.2.1 GeneralizedFareySequence

\[ \text{GeneralizedFareySequence}(fs) \]

Returns the generalized Farey sequence \( gfs \) of the Farey symbol.

Example

\[ \text{gap> GeneralizedFareySequence}(fs); \]
\[ [ \text{infinity}, 0, 1, 2, \text{infinity} ] \]

3.2.2 NumeratorOfGFSElement

\[ \text{NumeratorOfGFSElement}(gfs, i) \]

Returns: integer

Returns the numerator of the \( i \)-th term of the generalised Farey sequence \( gfs \): for the 1st infinite entry returns -1, for the last one returns 1, for all other entries returns the usual numerator.

Example

\[ \text{gap> List([1..5], i -> NumeratorOfGFSElement(GeneralizedFareySequence(fs),i));} \]
\[ [ -1, 0, 1, 2, 1 ] \]

3.2.3 DenominatorOfGFSElement

\[ \text{DenominatorOfGFSElement}(gfs, i) \]

Returns: integer

Returns the denominator of the \( i \)-th term of the generalised Farey sequence \( gfs \): for both infinite entries returns 0, for the other ones returns the usual denominator.
Example

```gap
List([1..5], i -> DenominatorGFSElement(GeneralizedFareySequence(fs),i));
[ 0, 1, 1, 1, 0 ]
```

3.2.4 LabelsOfFareySymbol

Returns the list of labels of the Farey symbol. This list has "odd", "even" and paired integers as entries.

Example

```gap
LabelsOfFareySymbol(fs);
[ 1, 2, 2, 1 ]
```
Chapter 4

Farey symbols for congruence subgroups

The package Congruence provides functions to construct Farey symbols for finite index subgroups. The algorithm used in the package allows to construct a Farey symbol for any finite index subgroup of $SL_2(\mathbb{Z})$ for which it is possible to check whether a given matrix belongs to this subgroup or not.

The development of an algorithm to determine the Farey symbol for a subgroup $G$ of a finite index in $SL_2(\mathbb{Z})$ was started by Ravi Kulkarni in [Kul91] and later it was improved by Mong-Lung Lang, Chong-Hai Lim and Ser-Peow Tan in [LLT95b], [LLT95a].

4.1 Computation of the Farey symbol for a finite index subgroup

4.1.1 FareySymbol

For a subgroup of a finite index $G$, this attribute stores one of the Farey symbols corresponding to the congruence subgroup $G$. The algorithm for its computation will work for any matrix group for which a membership test is available.

Example

```gap
gap> FareySymbol(PrincipalCongruenceSubgroup(8));
[ infinity, 0, 1/4, 1/3, 3/8, 2/5, 1/2, 3/5, 5/8, 2/3, 3/4, 1, 5/4, 4/3, 11/8, 7/5, 3/2, 8/5, 13/8, 5/3, 7/4, 2, 9/4, 7/3, 19/8, 12/5, 5/2, 13/5, 21/8, 8/3, 11/4, 3, 13/4, 10/3, 27/8, 17/5, 7/2, 18/5, 29/8, 11/3, 15/4, 4, 17/4, 13/3, 9/2, 14/3, 19/4, 5, 21/4, 16/3, 11/2, 17/3, 23/4, 6, 25/4, 19/3, 13/2, 20/3, 27/4, 7, 29/4, 22/3, 15/2, 23/3, 31/4, 8, infinity ]
[ 1, 17, 10, 26, 32, 18, 19, 27, 30, 5, 2, 2, 13, 28, 26, 20, 21, 29, 27, 7, 3, 3, 16, 31, 22, 23, 33, 29, 9, 4, 4, 5, 30, 31, 24, 25, 32, 33, 12, 6, 6, 7, 19, 18, 15, 8, 8, 9, 21, 20, 10, 11, 11, 12, 23, 22, 13, 14, 14, 15, 25, 24, 16, 17, 1 ]
gap> FareySymbol(CongruenceSubgroupGamma0(20));
[ infinity, 0, 1/5, 1/4, 2/7, 3/10, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 1, infinity ]
[ 1, 3, 4, 6, 7, 7, 5, 2, 2, 3, 6, 4, 5, 1 ]
```
4.2 Computation of generators of a finite index subgroup from its Farey symbol

If $fs$ is the Farey symbol for a group $G$ with $r_1$ even labels, $r_2$ odd labels and $r_3$ pairs of intervals, then $G$ is generated by $r_1 + r_2 + r_3$ matrices, which form a set of independent generators for $G$. These matrices are constructed as follows:

for each even interval $[x_i, x_{i+1}]$, take the matrix

$$A = \begin{pmatrix} a_{i+1}b_{i+1} + a_ib_i & -a_i^2 - a_{i+1}^2 \\ b_{i+1}^2 + b_i^2 & -a_{i+1}b_{i+1} - a_ib_i \end{pmatrix}$$

for each odd interval $[x_j, x_{j+1}]$, take the matrix

$$B = \begin{pmatrix} a_{j+1}b_{j+1} + a_jb_j + a_jb_j & -a_j^2 - a_{j+1}a_j - a_{j+1}^2 \\ b_{j+1}^2 + b_{j+1}b_j & -a_{j+1}b_{j+1} - a_jb_j - a_jb_j \end{pmatrix}$$

for each pair of free intervals $[x_k, x_{k+1}]$ and $[x_s, x_{s+1}]$, take the matrix

$$\begin{pmatrix} a_{s+1}b_{k+1} + a_sb_k & -a_sa_k - a_{s+1}a_{k+1} \\ b_{k+1}b_{k+1} & -a_{k+1}b_{s+1} - a_kb_s \end{pmatrix}$$

4.2.1 MatrixByEvenInterval

\textbf{Example}

```gap
gap> H:=CongruenceSubgroupGamma0(5);
<congruence subgroup CongruenceSubgroupGamma_0(5) in SL_2(Z)>
gap> fs:=FareySymbol(H);
[ infinity, 0, 1/2, 1, infinity ]
[ 1, "even", "even", 1 ]
gap> gfs:=GeneralizedFareySequence(fs);
[ infinity, 0, 1/2, 1, infinity ]
gap> MatrixByEvenInterval(gfs,2);
[ [ 2, -1 ], [ 5, -2 ] ]
```

4.2.2 MatrixByOddInterval

\textbf{Example}

```gap
gap> fs_oo:=FareySymbolByData([infinity,0,infinity],["odd","odd"]);
gap> gfs_oo:=GeneralizedFareySequence(fs_oo);
[ infinity, 0, infinity ]
gap> MatrixByOddInterval(gfs_oo,1);
[ [ -1, -1 ], [ 5, -2 ] ]
```
4.2.3 MatrixByFreePairOfIntervals

\( \text{MatrixByFreePairOfIntervals}(gfs, k, kp) \)

Returns the matrix corresponding to the pair of free intervals \( k \) and \( kp \) in the generalized Farey sequence \( gfs \).

Example

```gap
gap> fs_free:=FareySymbolByData([infinity,0,1,2,infinity],[1,2,1]);;
gap> gfs_free:=GeneralizedFareySequence(fs_free);;
gap> MatrixByFreePairOfIntervals(gfs_free,2,3);
[ [ 3, -2 ], [ 2, -1 ] ]
```

4.2.4 GeneratorsByFareySymbol

\( \text{GeneratorsByFareySymbol}(fs) \)

Returns a set of matrices constructed as above.

Example

```gap
gap> fs_eo:=FareySymbolByData([infinity,0,infinity],"even","odd");;
gap> GeneratorsByFareySymbol(last);
[ [ [ 0, -1 ], [ 1, 0 ] ], [ [ 0, -1 ], [ 1, -1 ] ] ]
gap> GeneratorsByFareySymbol(fs);
[ [ [ 1, 1 ], [ 0, 1 ] ], [ [ 2, -1 ], [ 5, -2 ] ], [ [ 3, -2 ], [ 5, -3 ] ] ]
gap> GeneratorsByFareySymbol(fs_oo);
[ [ [ -1, -1 ], [ 1, 0 ] ], [ [ 0, -1 ], [ 1, -1 ] ] ]
gap> GeneratorsByFareySymbol(fs_free);
[ [ [ 1, 2 ], [ 0, 1 ] ], [ [ 3, -2 ], [ 2, -1 ] ] ]
```

4.2.5 GeneratorsOfGroup

\( \text{GeneratorsOfGroup}(G) \)

Returns a set of generators for the finite index group \( G \) in \( SL_2(\mathbb{Z}) \).

Example

```gap
gap> G:=PrincipalCongruenceSubgroup(2);
<principal congruence subgroup of level 2 in SL_2(Z)>
gap> FareySymbol(G);
[ infinity, 0, 1, 2, infinity ]
[ 2, 1, 1, 2 ]
gap> GeneratorsOfGroup(G);
#I Using the Congruence package for GeneratorsOfGroup ...
[ [ [ 1, 2 ], [ 0, 1 ] ], [ [ 3, -2 ], [ 2, -1 ] ] ]
gap> H:=CongruenceSubgroupGamma0(5);
<congruence subgroup CongruenceSubgroupGamma_0(5) in SL_2(Z)>
gap> GeneratorsOfGroup(H);
```
Using the Congruence package for GeneratorsOfGroup ...

\[
\begin{bmatrix}
1, 1 \\
0, 1 \\
2, -1 \\
5, -2 \\
3, -2 \\
5, -3
\end{bmatrix}
\]

\[
\begin{bmatrix}
2, -1 \\
5, -2 \\
3, -2 \\
5, -3
\end{bmatrix}
\]

\[
\begin{bmatrix}
1, 2 \\
0, 1 \\
11, -2 \\
6, -1 \\
19, -8 \\
12, -5 \\
17, -10 \\
12, -7 \\
7, -6 \\
6, -5
\end{bmatrix}
\]

\[
\begin{bmatrix}
1, 5, 4, 3, 2, 2, 3, 4, 5, 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{infinity, 0, 1/3, 1/2, 2/3, 1, 4/3, 3/2, 5/3, 2, infinity}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{1, 5, 4, 3, 2, 2, 3, 4, 5, 1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{1, 5, 4, 3, 2, 2, 3, 4, 5, 1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{1, 5, 4, 3, 2, 2, 3, 4, 5, 1}
\end{bmatrix}
\]

4.3 Other properties derived from Farey symbols

4.3.1 IndexInPSL2ZByFareySymbol

> IndexInPSL2ZByFareySymbol(fs) (function)

By Proposition 7.2 in [Kulkarni], for the Farey symbol with underlying generalized Farey sequence \([\text{infinity, x0, x1, ..., xn, infinity}]\), the index in \(PSL_2(Z)\) is given by the formula \(d = 3n + e_3\), where \(e_3\) is the number of odd intervals.

Example

\[
\begin{bmatrix}
\text{IndexInPSL2ZByFareySymbol(fs);} \\
6
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{IndexInPSL2ZByFareySymbol(fs_oo);} \\
2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{IndexInPSL2ZByFareySymbol(fs_free);} \\
6
\end{bmatrix}
\]
Chapter 5

Service functions of the Congruence package

5.1 Additional information displayed by Congruence algorithms

5.1.1 InfoCongruence

InfoCongruence is a special Info class for Congruence algorithms. It has 3 levels: 0, 1 (default) and 2. To change the info level to k, use the command SetInfoLevel(InfoCongruence, k).

In the example below we use this mechanism to see more details during the Farey symbol construction for a congruence subgroup.
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