

# **Fwtree**

---

# **A GAP4 Package**

**Version 1.0**

**by**

**Bettina Eick**

ICM, TU Braunschweig, Germany

email: [beick@tu-bs.de](mailto:beick@tu-bs.de)

and

**Tobias Rossmann**

Department of Mathematics, NUI Galway, Ireland

email: [tobias.rossmann@gmail.com](mailto:tobias.rossmann@gmail.com)

**March 2018**

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Methods and functions</b>	<b>4</b>
2.1	Functions for finite $p$ -groups . . . . .	4
2.2	Functions to generate groups and trees . . . . .	4
2.3	Example . . . . .	5
	<b>Bibliography</b>	<b>7</b>
	<b>Index</b>	<b>8</b>

# 1

# Introduction

This package provides GAP-functions to reproduce the experimental results described in our paper [\[ER09\]](#).

More precisely, it provides

- functions to determine the rank, width and obliquity of a finite  $p$ -group,
- functions to investigate the graph of all finite  $p$ -groups of a given rank, width and obliquity using the ANUPQ-package [\[ONG06\]](#), and
- a library of finite quotients of certain infinite pro- $p$ -groups of finite rank, width and obliquity.

# 2

# Methods and functions

This chapter describes all the main methods and functions of this package.

## 2.1 Functions for finite $p$ -groups

Let  $G$  be a finite  $p$ -group given by a consistent polycyclic presentation as Pc group.

- 1 ▶ `LCSFactorTypes( G )`  
returns the abelian invariants of the lower central series factors of  $G$ .
- 2 ▶ `LCSFactorSizes( G )`  
returns the orders of the lower central series factors of  $G$ .
- 3 ▶ `WidthPGroup( G )`  
returns the width of  $G$ .
- 4 ▶ `SubgroupRank( G )`  
returns the (subgroup-)rank of  $G$ .
- 5 ▶ `Obliquity( G )`  
returns the obliquity of  $G$ .
- 6 ▶ `HasObliquityZero( G )`  
checks whether  $G$  has obliquity 0 and returns true or false.

## 2.2 Functions to generate groups and trees

Let  $G(p, rwo)$  denote the full tree of all finite  $p$ -groups with rank  $rwo[1]$ , width  $rwo[2]$  and obliquity  $rwo[3]$ . This tree can be finite or infinite; if it is infinite, then the infinite pro- $p$ -groups of the considered rank, width and obliquity specify infinite subtrees of the full tree. The groups not contained in such an infinite subtree are called sporadic.

- 1 ▶ `GroupsByRankWidthObliquity( p, d, rwo, roots, limit )`  
determines all  $p$ -groups  $G$  with  $G/\Phi(G)$  of order  $p^d$  and rank, width and obliquity as prescribed in  $rwo$  up to order  $limit$ . Here  $p$  and  $d$  are integers,  $rwo$  is a list of three integers and  $limit$  is an integer.  
The parameter  $roots$  is a list of groups described by their id's with respect to the small groups library. The descendants of the groups described in  $roots$  are excluded from the output of this function. This option can be used to prune the tree of groups determined by this function.  
If there are only finitely many sporadic  $p$ -groups with given rank, width and obliquity, then this function can be used to generate them; in this case  $roots$  must contain a complete list of all id's of roots of infinite subtrees and  $limit$  can be set to infinity.

2 ▶ BranchRWO(  $G$ ,  $i$ ,  $rwo$  )

for a stable quotient (see [ER09])  $G$  of a pro- $p$ -group of rank  $rwo[1]$ ,  $rwo[2]$  and obliquity  $rwo[3]$ , this function returns the  $i$ -th branch of its corresponding tree. The structure of the tree is encoded in a list. If one of the global parameters CHECK\_RANK or CHECK\_OBLIQUITY is set to false, then checking the corresponding invariant is omitted and hence a potentially larger tree is returned.

The user is advised not to perform any other computations using ANUPQ or the pq-program while using this or the following function, because such computations will be terminated.

3 ▶ BoundedDescendantsRWO(  $G$ ,  $i$ ,  $c$ ,  $rwo$  )

returns the tree of all descendants of  $G/\gamma_i(G)$  of rank  $rwo[1]$ , width  $rwo[2]$ , obliquity  $rwo[3]$  and class at most  $c$ .

4 ▶ DrawBranch(  $branch$  )

if the package is run under XGap, then this function can be used to draw a branch as output by the above two functions in the case of width 2. The user may wish to improve the quality of the output by modifying the file `gap/xbranch.gi`.

Vertices drawn on the same level correspond to groups of the same class. If  $G$  is a descendant of  $H$  in the branch, then  $G$  is drawn as a filled circle if  $|G| = |H|p$  and as a solid box if  $|G| = |H|p^2$ .

The package also provides finite quotients of a number of infinite pro- $p$ -groups with finite rank, width and obliquity. Throughout the section,  $p$  is an odd prime.

5 ▶ ProPSylowGroupOfPSL( $d$ ,  $p$ ,  $n$ )

returns the quotient of the Sylow pro- $p$ -subgroup of  $PSL_d(\mathbb{Q}_p)$  modulo the matrices which are congruent to the identity modulo  $p^n$ .

6 ▶ ProPSylowGroupOfPSF( $p$ ,  $n$ )

Let  $L$  be the simple Lie algebra of dimension 3 over  $\mathbb{Q}_p$  which is not isomorphic to  $sl_2(\mathbb{Q}_p)$ . This function returns a finite quotient of the Sylow pro- $p$ -subgroup of its automorphism group. The parameter  $n$  specifies how large this quotient is.

In [KLG97], a library of maximal pro- $p$ -groups with finite rank, width and obliquity corresponding to the Lie algebras of small dimension is provided. Here, we provide a library of large quotients of these groups for some of the Lie algebras of type  $sl_d(K)$ , where  $K$  is a finite extension of  $\mathbb{Q}_p$ . These groups have been determined using the programs described in [KLG97]. To be precise, depending on the group, it may be necessary to pass to the quotient by one of the last non-trivial terms of the lower central series in order to obtain a quotient of the respective pro- $p$ -group.

7 ▶ ProPQuotient( $p$ ,  $dim$ ,  $deg$ ,  $no$ )

returns a finite group corresponding to the maximal pro- $p$ -group  $G$  with Lie algebra  $sl_{dim}(K)$ , where  $K$  is a field of degree  $deg$  over  $\mathbb{Q}_p$ . The parameter  $no$  specifies the number of the group in our database.

## 2.3 Example

When run under XGap, the following code constructs and draws the branch with root  $G/\gamma_5(G)$  in the graph of finite 5-groups of rank 3, width 2 and obliquity 0, where  $G$  is the Sylow pro- $p$ -subgroup of  $\text{Aut}(sl_2(\mathbb{Q}_5))$ .

```
gap> g := ProPSylowGroupOfPSL(2,5,6);
Pcp-group with orders [ 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5 ]
gap> branch := BranchRWO(g,5,[3,2,0]);;
ConstructBranch: root-p-class: 4
Constructed 3 1-step descendants.
ConstructBranch: root-p-class: 5
Constructed 0 1-step descendants.
Constructed 0 2-step descendants.
ConstructBranch: root-p-class: 5
Constructed 0 1-step descendants.
Constructed 0 2-step descendants.
ConstructBranch: root-p-class: 5
Constructed 0 1-step descendants.
Constructed 0 2-step descendants.
Constructed 3 2-step descendants.
ConstructBranch: root-p-class: 5
Constructed 0 1-step descendants.
Constructed 0 2-step descendants.
ConstructBranch: root-p-class: 5
Constructed 0 1-step descendants.
Constructed 0 2-step descendants.
time: 0:00:16.525
gap> DrawBranch(branch);
```

A window with the following graph should appear.

# Bibliography

- [ER09] Bettina Eick and Tobias Rossmann. Periodicities for graphs of  $p$ -groups beyond coclass. 2009. Preprint.
- [KLG97] G. Klaas, C. R. Leedham-Green, and W. Plesken. *Linear pro- $p$ -groups of finite width*, volume 1674 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997.
- [ONG06] Eamonn O'Brien, Werner Nickel, and Greg Gamble. *ANUPQ — A GAP4 Package, Version 3.0*, 2006.

# Index

This index covers only this manual. A page number in *italics* refers to a whole section which is devoted to the indexed subject. Keywords are sorted with case and spaces ignored, e.g., “PermutationCharacter” comes before “permutation group”.

## **B**

BoundedDescendantsRWO, 5  
BranchRWO, 5

## **D**

DrawBranch, 5

## **E**

Example, 5

## **F**

Functions for finite  $p$ -groups, 4  
Functions to generate groups and trees, 4

## **G**

GroupsByRankWidthObliquity, 4

## **H**

HasObliquityZero, 4

## **L**

LCSFactorSizes, 4  
LCSFactorTypes, 4

## **O**

Obliquity, 4

## **P**

ProPQuotient, 5  
ProPSylowGroupOfPSF, 5  
ProPSylowGroupOfPSL, 5

## **S**

SubgroupRank, 4

## **W**

WidthPGroup, 4