SglPPow

—

A GAP4 Package

Version 2.0

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Contents

1 Introduction

SglPPow is a package which extends the Small Groups Library. Currently the Small Groups Library gives access to the following groups:

(1)
Those of order at most 2000 except 1024 (423,164,062 groups);

(2)
Those of cubefree order at most 50,000 (395,703 groups);

(3)
Those of order p^7 for the primes $p = 3, 5, 7, 11$ (907,489 groups);

(4)
Those of order p^n for $n \leq 6$ and all primes p;

(5)
Those of order pq^n where q^n divides 28, 36, 55 or 74 and p is an arbitrary prime not equal to q;

(6)
Those of squarefree order;

(7)
Those whose order factorizes into at most 3 primes.

This package gives access to the groups of order p^7 for primes $p > 11$, and to the groups of order 3^8.

To access the groups of order p^7 for primes $p > 11$ you need the packages LiePRing (by Michael Vaughan-Lee and Bettina Eick) and LieRing (by Willem de Graaf and Serena Cicalo).

The groups of order 3^8 have been determined by Michael Vaughan-Lee. The groups of order p^7 for primes $p > 11$ are available via the database of the nilpotent Lie rings of order p^k for $k \leq 7$ and primes $p > 3$ in the LiePRing package. These groups are obtained from the Lie rings using the implementation of the Baker-Campbell-Hausdorff formula in the LieRing package.

Acknowledgements: The authors thank Max Horn for help with general framework of GAP programs to extend the Small Groups Library.

2 Accessing the data

When this package is loaded, then the groups of order 3^8 and p^7 for primes $p > 11$ are additionally available via the SmallGroups library. As a result, all groups of order p^n with $p = 2$ and $n \leq 9$ and $p = 3$ and $n \leq 8$ and p an arbitrary prime and $n \leq 7$ are then available via the small groups library. The corresponding information can be obtained via

1 ► `SmallGroup(size, number)`

2 ► `NumberSmallGroups(size)`

3 ► `SmallGroupsInformation(size)`

See Section 50.7 in the GAP manual for background on these functions. Note that there is no IdGroup function available for this extension of the small groups library.

WARNING: The user should be aware that there are there are 1,396,077 groups of order 3^8, 1,600,573 groups of order 13^7, and 5,546,909 groups of order 17^7. For general p the number of groups of order p^7 is a PORC polynomial in p with leading term $3p^5$. Furthermore, as the prime p increases, the time taken to generate a complete list of the groups of order p^7 grows rapidly. Experimentally the time seems to be proportional to $p^{6\cdot 2}$. For $p = 13$ it takes several hours to generate the complete list. For primes $p \leq 11$ the groups are precomputed, and their SmallGroup codes are stored in the SmallGroups database. For primes $p > 11$ the Lie rings have to be generated from 4773 parametrized presentations in the LiePRing database, and then converted into groups using the Baker-Campbell-Hausdorff formula. A complete list of power commutator presentations for the groups of order 13^7 takes over 11 gb of memory.

```
gap> NumberSmallGroups(3^8);
1396077
gap> SmallGroup(3^8, 1000000);
<pc group of size 6561 with 8 generators>
gap> NumberSmallGroups(17^7);
5546909
gap> SmallGroup(17^7, 5000);
constructing a batch of 1156 groups ... this may take a while
<pc group of size 410338673 with 7 generators>
gap> NumberSmallGroups(101^7);
32826263845
```

3 The organization of the data

We include some brief comments on the organization of the data. As a preliminary step we briefly recall the p-group generation algorithm.

3.1 The p-group generation algorithm

The p-group generation algorithm was developed and implemented by Eamonn O'Brien, and we refer the reader to [O'B90] for a detailed description of the algorithm.

For a brief overview, let P be a p-group. The algorithm uses the lower p-central series, defined recursively by $\lambda_1(P) = P$ and $\lambda_{i+1}(P) = [\lambda_i(P), P]\lambda_i(P)^p$ for $i \geq 1$. The p-class of P is the length of this series. Each p-group P, apart from the elementary abelian ones, is an immediate descendant of the quotient P/R where R is the last non-trivial term of the lower p-central series of P.

Thus all the groups with order 3^8, except the elementary abelian one, are immediate descendants of groups with order 3^k for some k smaller than 8. All of the immediate descendants of a p-group Q are quotients of a certain extension of Q; the isomorphism problem for these descendants is equivalent to the problem of determining orbits of certain subgroups of this extension under an action of the automorphism group of Q. Not all p-groups have immediate descendants, those that do are called capable, and those which do not are called terminal.

O'Brien and Vaughan-Lee's classification of the groups of order p^7 in [OVL05] is based on a classification of the nilpotent Lie rings of order p^7, and the groups of order p^7 are obtained from the Lie rings using the Baker-Campbell-Hausdorff formula. O'Brien and Vaughan-Lee classified the nilpotent Lie rings of order p^7 using the nilpotent Lie ring generation algorithm, which is a direct analogue of the p-group generation algorithm.

Thus the databases of nilpotent Lie rings of order p^7 and of the groups of order 3^8 are organized according to these algorithms: the immediate descendants of order p^7 of each nilpotent Lie ring of order less than p^7 are grouped together in the database of nilpotent Lie rings, and the immediate descendants of order 3^8 of each group of order less than 3^8 are grouped together in the database of groups of order 3^8.

3.2 The groups of order 6561

The database of groups of order 3^8 is organized according to rank and p-class. Here rank is the rank of the Frattini quotient, i.e. the size of a minimal generating set, and p-class is as defined in the previous section. The following table gives the number of groups of order 3^8 of each rank and p-class, with the (i, j) entry corresponding to rank i and p-class j.

	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	1
2	0	0	58	486	1343	330	9	0
3	0	4	216747	40521	2163	24	0	0
4	0	23361	494666	22343	51	0	0	0
5	0	578478	14796	80	0	0	0	0
6	0	566	39	0	0	0	0	0
7	0	10	0	0	0	0	0	0
8	1	0	0	0	0	0	0	0

In the list of all groups of order 3^8 the first group is the cyclic group, then the 2-generator groups follow in order of increasing p-class and so on. The above table can thus be used to find the range of numbers of groups with a given rank and p-class.

As mentioned above, the database is organized according to the p-group generation algorithm. For example, the 9 groups of rank 2, p-class 7, and order 3^8 are numbered from 2219–2227. The groups numbered 2219 and 2220 are descendants of SmallGroup($3^7, 384$), and the groups numbered 2221–2227 are descendants of SmallGroup($3^7, 386$). Similarly, the 24 groups of rank 3, p-class 6, and order 3^8 are numbered from 261663–261686. The first four of these groups are descendants of SmallGroup($3^7, 5841$), the next 17 are descendants of SmallGroup($3^7, 5844$), and the last 4 are descendants of SmallGroup($3^7, 5849$).

3.3 The groups with order the seventh power of a prime

The groups of order p^7 for prime $p > 11$ are obtained from the LiePRing database of nilpotent Lie rings of order p^7 using Willem de Graaf's implementation of the Baker-Campbell-Hausdorff formula. The LiePRing database is organized according to the output from the nilpotent Lie ring generation algorithm. For any given p the first Lie ring in the database is the cyclic Lie ring of order p^7. Next come the two generator Lie rings, then the three generator Lie rings, and so on, ending with the six generator Lie rings, and then finally the elementary abelian Lie ring of rank 7. The first four of the two generator nilpotent Lie rings of order p^7 are immediate descendants of the Lie ring

$$\langle a, b \mid pb,\ class 3\rangle$$

of order p^4. The next $p^2 + 8p + 25$ are immediate descendants of the Lie ring

$$\langle a, b \mid baa, bab, pba,\ class 3\rangle$$

of order p^5, and the next $p + 6 + (p^2 + 3p + 10)gcd(p - 1, 3)$ are immediate descendants of

$$\langle a, b \mid babb, pa, pb,\ class 4\rangle\cdot$$

The nine rank 6 Lie rings in the database are the rank 6, p-class 2 Lie rings. These are the immediate descendants of the elementary abelian Lie ring of rank 6.

There is a complete list of presentations for the nilpotent Lie rings of order p^k for $k \leq 7$ valid for all $p > 3$ in the document p567.pdf supplied with the documentation for the LiePRing package. The presentations are grouped as described above, with each group of presentations giving the immediate descendants of a Lie ring of smaller order.

In a few cases the descendants of a parametrized family of Lie rings are grouped together. For example there is a family of $p(p - 1)$ distinct Lie rings with presentations of the form

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - \lambda baa - \mu bab,\ pb + \nu baa + \xi bab,\ pc,\ class 3\rangle$$

with $\lambda, \mu, \nu, \xi \neq 0$. Most of these algebras are terminal, but $\frac{5}{2}p - \frac{9}{2} + \frac{1}{2}\gcd(p-1,4)$ of them are capable, and they have a total of $\frac{1}{2}p^3 + 2p^2 - 5p + \frac{1}{2} + \frac{p}{2}\gcd(p-1,4)$ descendants of order p^7 and p-class 4. These descendants have presentations

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \mu bab - ybaaa,\ pb + \nu baa + bab - zbaaa,\ pc - tbaaa,\ class4\rangle,$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \mu bab - ybaaa,\ pb + \nu baa + \mu\nu bab - zbaaa,\ pc - tbaaa,\ class4\rangle$$

for various choices of the parameters μ, ν, y, z, t. For any given value of p the $\frac{1}{2}p^3+2p^2-5p+\frac{1}{2}+\frac{p}{2}\gcd(p-1,4)$ distinct Lie algebras with presentations of this form are grouped together, with consecutive numbering.

There is no easy way to determine the numbering of (say) the three generator Lie rings of p-class 4 since the numbers depend on p in a very complicated way, and generally there is no easy efficient way of searching the database for a group with particular properties. In view of the numbers of groups of order p^7 and the time needed to generate a complete list, this means that the database will be of limited use for most people. A user who wants to access a particular batch of descendants as described in p567.pdf is advised to use the LiePRing package directly, as this package also has an option to obtain the corresponding groups via Willem de Graaf's implementation of the Baker-Campbell-Hausdorff formula. On the other hand, there are only

$$2p^3 + 13p^2 + 64p + 145 + (p^2 + 10p + 56)gcd(p-1,3) + (4p + 28)gcd(p-1,4)$$
$$+(2p + 12)gcd(p-1,5) + gcd(p-1,7) + 4gcd(p-1,8) + gcd(p-1,9)$$

two generator groups of order p^7 for $p > 5$ and a complete list of them can be generated quite quickly for moderate values of p. The table below gives the number of d generator groups of order p^7 valid for all $p > 5$, and the user can use the table to compute the range of numbers needed to access the d generator groups of order p^7 for any given p.

rank 1

1

rank 2

$2p^3 + 13p^2 + 64p + 145 + (p^2 + 10p + 56)gcd(p-1,3) + (4p + 28)gcd(p-1,4)$
$+(2p + 12)gcd(p-1,5) + gcd(p-1,7) + 4gcd(p-1,8) + gcd(p-1,9)$

rank 3

$2p^5 + 9p^4 + 29p^3 + 99p^2 + 380p + 1100 + (3p^2 + 28p + 189)gcd(p-1,3)$
$+(p^2 + 13p + 84)gcd(p-1,4) + (p + 17)gcd(p-1,5) + gcd(p-1,8) + 3gcd(p-1,7)$

rank 4

$p^5 + 3p^4 + 13p^3 + 57p^2 + 248p + 1044 + (6p + 46)gcd(p-1,3)$
$+(2p + 23)\gcd(p-1,4) + 2\gcd(p-1,5)$

rank 5

$p^2 + 15p + 155$

rank 6

9

rank 7

1

The total number of groups of order p^7 for all $p > 5$ is given by

$$3p^5 + 12p^4 + 44p^3 + 170p^2 + 707p + 2455$$
$$+(4p^2 + 44p + 291)gcd(p-1,3) + (p^2 + 19p + 135)gcd(p-1,4)$$
$$+(3p + 31)gcd(p-1,5) + 4gcd(p-1,7) + 5gcd(p-1,8) + gcd(p-1,9)\cdot$$

4 Installing and loading the package

The package can be downloaded from www.icm.tu-bs.de/ beick/soft/sglppow/sglppow.tar

Then tar -zxvf slgppow.tar produces a directory sglppow. This should be moved into the pkg directory of a GAP installation.

The package is set up so that after loading it into GAP with LoadPackage the full functionality of the package is available.

Bibliography

[O'B90] Eamonn O'Brien. The p-group generation algorithm. *J. Symbolic Computation*, 9:677–698, 1990.

[OVL05] Eamonn A. O'Brien and Michael R. Vaughan-Lee. The groups with order p^7 for odd prime p. *J. Algebra*, 292(1):243–258, 2005.