Unipot

A GAP4 Package

For Computations with elements of unipotent subgroups of Chevalley Groups

by

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1 Preface
   1.1 Root Systems ................................................................. 3
   1.2 Citing Unipot ................................................................. 3
2 The GAP Package Unipot
   2.1 General functionality ...................................................... 4
   2.2 Unipotent subgroups of Chevalley groups ............................. 4
   2.3 Elements of unipotent subgroups of Chevalley groups ............. 7
   2.4 Symbolic computation ..................................................... 12
   Bibliography ..................................................................... 13
   Index ............................................................................. 14
Unipot is a package for GAP4 [GAP04]. The version 1.0 of this package was the content of my diploma thesis [Hal00].

Let $U$ be a unipotent subgroup of a Chevalley group of Type $L(K)$. Then it is generated by the elements $x_r(t)$ for all $r \in \Phi^+, t \in K$. The roots of the underlying root system $\Phi$ are ordered according to the height function. Each element of $U$ is a product of the root elements $x_r(t)$. By Theorem 5.3.3 from [Car89] each element of $U$ can be uniquely written as a product of root elements with roots in increasing order. This unique form is called the canonical form.

The main purpose of this package is to compute the canonical form of an element of the group $U$. For we have implemented the unipotent subgroups of Chevalley groups and their elements as GAP objects and installed some operations for them. One method for the operation $\text{Comm}$ uses the Chevalley’s commutator formula, which we have implemented, too.

1.1 Root Systems

We are using the root systems and the structure constants available in GAP from the simple Lie algebras. We also are using the same ordering of roots available in GAP.

1.2 Citing Unipot

If you use Unipot to solve a problem or publish some result that was partly obtained using Unipot, I would appreciate it if you would cite Unipot, just as you would cite another paper that you used. (Below is a sample citation.) Again I would appreciate if you could inform me about such a paper.

Specifically, please refer to:


(Should the reference style require full addresses please use: “Arbeitsgruppe Algebra, Mathematisches Institut, Justus-Liebig-Universität Gießen, Arndtstr. 2, 35392 Gießen, Germany”)
This chapter describes the package 

2Unipot

Mainly, the package provides the ability to compute with elements of unipotent subgroups of Chevalley groups, but also some properties of this groups.

In this chapter we will refer to unipotent subgroups of Chevalley groups as “unipotent subgroups” and to elements of unipotent subgroups as “unipotent elements”. Specifically, we only consider unipotent subgroups generated by all positive root elements.

2.1 General functionality

In this section we will describe the general functionality provided by this package.

\[\text{UnipotChevInfo}\]

UnipotChevInfo is an InfoClass used in this package. InfoLevel of this InfoClass is set to 1 by default and can be changed to any level by \(\text{SetInfoLevel( UnipotChevInfo, n )}\).

Following levels are used throughout the package:

1. —
2. When calculating the order of a finite unipotent subgroup, the power presentation of this number is printed. (See 2.2.5 for an example)
3. When comparing unipotent elements, output, for which of them the canonical form must be computed. (See 2.3.11 for an example)
4. —
5. While calculating the canonical form, output the different steps.
6. The process of calculating the Chevalley commutator constants is printed on the screen

2.2 Unipotent subgroups of Chevalley groups

In this section we will describe the functionality for unipotent subgroups provided by this package.

\[\text{IsUnipotChevSubGr( grp )}\]

Category for unipotent subgroups.

\[\text{UnipotChevSubGr( type, n, F )}\]

UnipotChevSubGr returns the unipotent subgroup \(U\) of the Chevalley group of type \(type\), rank \(n\) over the ring \(F\).

type must be one of "A", "B", "C", "D", "E", "F", "G".
For the type "A", \(n\) must be a positive integer.
For the types "B" and "C", \(n\) must be a positive integer \(\geq 2\).
For the type "D", \(n\) must be a positive integer \(\geq 4\).
For the type "E", \(n\) must be one of 6, 7, 8.
Section 2. Unipotent subgroups of Chevalley groups

For the type "F", $n$ must be 4.
For the type "G", $n$ must be 2.

```gap
gap> U_G2 := UnipotChevSubGr("G", 2, Rationals);
<Unipotent subgroup of a Chevalley group of type G2 over Rationals>
gap> IsUnipotChevSubGr(U_G2);
true
gap> UnipotChevSubGr("E", 3, Rationals);
Error, <n> must be one of 6, 7, 8 for type E called from
UnipotChevFamily( type, n, F ) called from
<function>( <arguments> ) called from read-eval-loop
Entering break read-eval-print loop ...
you can 'quit;' to quit to outer loop, or
you can 'return;' to continue
brk>
```

Special methods for unipotent subgroups. (see GAP Reference Manual, section 6.3 for general information on View and Print)

```gap
gap> Print(U_G2);
UnipotChevSubGr( "G", 2, Rationals )gap> View(U_G2);
<Unipotent subgroup of a Chevalley group of type G2 over Rationals>
```

Special methods for unipotent subgroups. Return the identity element of the group $U$. The returned element has representation UNIPOT_DEFAULT_REP (see 2.3.3).

```gap
gap> One(U_G2);
```

Special methods for unipotent subgroups. This is a special method for unipotent subgroups using the result in Carter [Car89], Theorem 5.3.3 (ii).

```gap
gap> Size(UnipotChevInfo, 2);
gap> Size(UnipotChevSubGr("E", 8, GF(7)));
#I The order of this group is 7^120 which is
25808621098934927604791781741317238363169114027609954791128059842592785343731\ 
743726320645695945672001
gap> SetInfoLevel(UnipotChevInfo, 1);
```

This method is similar to the method RootSystem for semisimple Lie algebras (see Section 64.6 in the GAP Reference Manual for further information).

RootSystem returns the underlying root system of the unipotent subgroup $U$. The returned object is from the category IsRootSystem:
Chapter 2. The GAP Package Unipot

```gap
R_G2 := RootSystem(U_G2);
<root system of rank 2>
IsRootSystem(last);
true
SimpleSystem(R_G2);
[[2, -1], [-3, 2]]
```

Additionally to the properties and attributes described in the Reference Manual, following attributes are installed for the Root Systems by the package Unipot:

1. **PositiveRootsFC(R)**
   - The list of positive roots of the root system $R$. Every root is represented as a list of coefficients of the linear combination in fundamental roots. E.g. let $r = \sum_{i=1}^{l} k_i r_i$, where $r_1, \ldots, r_l$ are the fundamental roots, then $r$ is represented as the list $[k_1, \ldots, k_l]$.

2. **NegativeRootsFC(R)**
   - This is a special Method for unipotent subgroups of finite Chevalley groups.

3. **GeneratorsOfGroup(U)**
   - This method returns an element of the unipotent subgroup $U$ with indeterminates instead of ring elements. Such an element could be used for symbolic computations (see 2.4). The returned element has representation UNIPOT_DEFAULT REP (see 2.3.3).

4. **CentralElement(U)**
   - This method returns the representative of the center of $U$ without calculating the center.
2.3 Elements of unipotent subgroups of Chevalley groups

In this section we will describe the functionality for unipotent elements provided by this package.

1 ▶ IsUnipotChevElem( elm )
Category for elements of a unipotent subgroup.

2 ▶ IsUnipotChevRepByRootNumbers( elm )
▶ IsUnipotChevRepByFundamentalCoeffs( elm )
▶ IsUnipotChevRepByRoots( elm )

IsUnipotChevRepByRootNumbers, IsUnipotChevRepByFundamentalCoeffs and IsUnipotChevRepByRoots are different representations for unipotent elements.

Roots of elements with representation IsUnipotChevRepByRootNumbers are represented by their numbers (positions) in PositiveRoots(RootSystem(U)).

Roots of elements with representation IsUnipotChevRepByFundamentalCoeffs are represented by elements of PositiveRootsFC(RootSystem(U)).

Roots of elements with representation IsUnipotChevRepByRoots are represented by roots themselves, i.e. elements of PositiveRoots(RootSystem(U)).

(See 2.3.4, 2.3.5 and 2.3.6 for examples.)

3 ▶ UNIPOT_DEFAULT_REP

This variable contains the default representation for newly created elements, e.g. created by One or Random. When Unipot is loaded, the default representation is IsUnipotChevRepByRootNumbers and can be changed by assigning a new value to UNIPOT_DEFAULT_REP.

    gap> UNIPOT_DEFAULT_REP := IsUnipotChevRepByFundamentalCoeffs;;

Note that Unipot doesn’t check the type of this value, i.e. you may assign any value to UNIPOT_DEFAULT_REP, which may result in errors in following commands:

    gap> UNIPOT_DEFAULT_REP := 3;;
    gap> One( U_G2 );
    ... Error message ...

4 ▶ UnipotChevElemByRootNumbers( U, roots, felems )
▶ UnipotChevElemByRootNumbers( U, root, felem )
▶ UnipotChevElemByRN( U, roots, felems )
▶ UnipotChevElemByRN( U, root, felem )

UnipotChevElemByRootNumbers returns an element of a unipotent subgroup U with representation IsUnipotChevRepByRootNumbers (see 2.3.2).

roots should be a list of root numbers, i.e. integers from the range 1, ..., Length(PositiveRoots(RootSystem(U))).

And felems a list of corresponding ring elements or indeterminates over that ring (see GAP Reference Manual, 66.1.1 for general information on indeterminates or section 2.4 of this manual for examples).

The second variant of UnipotChevElemByRootNumbers is an abbreviation for the first one if roots and felems contain only one element.

UnipotChevElemByRN is just a synonym for UnipotChevElemByRootNumbers.
gap> IsIdenticalObj( UnipotChevElemByRN, UnipotChevElemByRootNumbers );
true

In this example we create two elements: \( x_{r_1}(2) \cdot x_{r_5}(7) \) and \( x_{r_1}(2) \), where \( r_i \), \( i = 1, \ldots, 6 \) are the positive roots in \( \text{PositiveRoots}(\text{RootSystem}(U)) \) and \( x_{r_i}, i = 1, \ldots, 6 \) the corresponding root elements.

UnipotChevElemByFundamentalCoeffs returns an element of a unipotent subgroup \( U \) with representation \( \text{IsUnipotChevRepByFundamentalCoeffs} \) (see 2.3.2).

\( \text{roots} \) should be a list of elements of \( \text{PositiveRootsFC}(\text{RootSystem}(U)) \). And \( \text{felems} \) a list of corresponding ring elements or indeterminates over that ring (see \text{GAP Reference Manual}, 66.1.1 for general information on indeterminates or section 2.4 of this manual for examples).

The second variant of \( \text{UnipotChevElemByFundamentalCoeffs} \) is an abbreviation for the first one if \( \text{roots} \) and \( \text{felems} \) contain only one element.

UnipotChevElemByFC is just a synonym for \( \text{UnipotChevElemByFundamentalCoeffs} \).

In this example we create the same two elements as in 2.3.4: \( x_{\{1,0\}}(2) \cdot x_{\{3,1\}}(7) \) and \( x_{\{1,0\}}(2) \), where \( \{1,0\} = r_1 + 0r_2 = r_1 \) and \( \{3,1\} = 3r_1 + 1r_2 = r_5 \) are the first and the fifth positive roots of \( \text{PositiveRootsFC}(\text{RootSystem}(U)) \) respectively.

UnipotChevElemByRoots returns an element of a unipotent subgroup \( U \) with representation \( \text{IsUnipotChevRepByRoots} \) (see 2.3.2).

\( \text{roots} \) should be a list of elements of \( \text{PositiveRoots}(\text{RootSystem}(U)) \). And \( \text{felems} \) a list of corresponding ring elements or indeterminates over that ring (see \text{GAP Reference Manual}, 66.1.1 for general information on indeterminates or section 2.4 of this manual for examples).

The second variant of \( \text{UnipotChevElemByRoots} \) is an abbreviation for the first one if \( \text{roots} \) and \( \text{felems} \) contain only one element.

UnipotChevElemByR is just a synonym for \( \text{UnipotChevElemByRoots} \).
Section 3. Elements of unipotent subgroups of Chevalley groups

gap> PositiveRoots(RootSystem(U_G2));
[ [ 2, -1 ], [ -3, 2 ], [ -1, 1 ], [ 1, 0 ], [ 3, -1 ], [ 0, 1 ] ]
gap> y2 := UnipotChevElemByRoots( U_G2, [[ 2, -1 ], [ 3, -1 ]], [2,7] );
x_{[ 2, -1 ]}( 2 ) * x_{[ 3, -1 ]}( 7 )
gap> x2 := UnipotChevElemByRoots( U_G2, [ 2, -1 ], 2 );
x_{[ 2, -1 ]}( 2 )
In this example we create again the two elements as in previous examples: $x_{[2,-1]}(2) \cdot x_{[3,-1]}(7)$ and $x_{[2,-1]}(2)$, where $[2,-1] = r_1$ and $[3,-1] = r_5$ are the first and the fifth positive roots of $\text{PositiveRoots}(\text{RootSystem}(U))$ respectively.

These three methods are provided for converting a unipotent element to the respective representation. If $x$ has already the required representation, then $x$ itself is returned. Otherwise a new element with the required representation is generated.

gap> x;
x_{[1]}(2)
gap> x1 := UnipotChevElemByFundamentalCoeffs( x );
x_{[1,0]}(2)
gap> IsIdenticalObj(x, x1); x = x1;
false
gap> x2 := UnipotChevElemByFundamentalCoeffs( x1 );;
gap> IsIdenticalObj(x1, x2);
true

Note: If some attributes of $x$ are known (e.g. Inverse (see 2.3.15) or CanonicalForm (see 2.3.8)), then they are “converted” to the new representation, too.

DEPRECATED These are old versions of UnipotChevElemByXX (from Unipot 1.0 and 1.1). They are deprecated now and exist for compatibility only. They may be removed at any time.

CanonicalForm returns the canonical form of $x$. For more information on the canonical form see Carter [Car89], Theorem 5.3.3 (ii). It says:

Each element of a unipotent subgroup $U$ of a Chevalley group with root system $\Phi$ is uniquely expressible in the form

$$ \prod_{r_i \in \Phi^+} x_{r_i}(t_i), $$

where the product is taken over all positive roots in increasing order.

gap> z := UnipotChevElemByFC( U_G2, [[0,1], [1,0]], [3,2] );
x_{[0,1]}(3) * x_{[1,0]}(2)
gap> CanonicalForm(z);
x_{[1,0]}(2) * x_{[0,1]}(3) * x_{[1,1]}(6) * x_{[2,1]}(12) * x_{[3,1]}(24) * x_{[3,2]}(-72)
So if we call the positive roots $r_1, \ldots, r_6$, we have $z = x_{r_1}(3)x_{r_2}(2) = x_{r_1}(2)x_{r_2}(3)x_{r_3}(6)x_{r_4}(12)x_{r_5}(24)x_{r_6}(-72)$.

Special methods for unipotent elements. (see GAP Reference Manual, section 6.3 for general information on View and Print). The output depends on the representation of $x$.

```
gap> Print(x);
UnipotChevElemByRootNumbers( UnipotChevSubGr( "G", 2, Rationals ), \[ 1 ], [ 2 ] )gap>
View(x);
x_{1}( 2 )gap>
```

This is a special method for unipotent elements. 

ShallowCopy creates a copy of $x$. The returned object is not identical to $x$ but it is equal to $x$ w.r.t. the equality operator $=$. Note that CanonicalForm and Inverse of $x$ (if known) are identical to CanonicalForm and Inverse of the returned object.

(See GAP Reference Manual, section 12.7 for further information on copyability)

Special method for unipotent elements. If $x$ and $y$ are identical or are products of the same root elements then true is returned. Otherwise CanonicalForm (see 2.3.8) of both arguments must be computed (if not already known), which may be expensive. If the canonical form of one of the elements must be calculated and InfoLevel of UnipotChevInfo is at least 3, the user is notified about this:

```
gap> y := UnipotChevElemByRN( U_G2, [1,5], [2,7] );
x_{1}( 2 ) * x_{5}( 7 )gap>
```

Special Method for UnipotChevElem

This is needed e.g. by AsSSortetList.

The ordering is computed in the following way: Let $x = x_{r_1}(s_1) \cdots x_{r_n}(s_n)$ and $y = x_{r_1}(t_1) \cdots x_{r_n}(t_n)$, then

$$x < y \iff [s_1, \ldots, s_n] < [t_1, \ldots, t_n],$$
where the lists are compared lexicographically. e.g. for $x = x_r(1)x_r(1) = x_r(1)x_r(1)x_r(0)$ (field elems: [ 1, 1, 0 ]) and $y = x_r(1)x_r(1) = x_r(1)x_r(0)x_r(1)$ (field elems: [ 1, 0, 1 ]) we have $y < x$ (above lists ordered lexicographically).

Special method for unipotent elements. The expressions in the form $x_r(t)x_r(u)$ will be reduced to $x_r(t + u)$ whenever possible.

```
gap> y; z;
x_{1}( 2 ) * x_{5}( 7 )
x_{5}( 7 ) * x_{1}( 2 )
gap> y * z;
x_{1}( 2 ) * x_{5}( 14 ) * x_{1}( 2 )
```

Note: The representation of the product will be always the representation of the first argument.

```
gap> x; x1; x = x1;
x_{1}( 2 )
x_{[ 1, 0 ]}( 2 )
true
gap> x * x1;
x_{1}( 4 )
gap> x1 * x;
x_{[ 1, 0 ]}( 4 )
```

Special method for unipotent elements. $0_{e0p}$ returns the multiplicative neutral element of $x$. This is equal to $x^0$.

Special methods for unipotent elements. We are using the fact

$$
\left( x_r(t_1) \cdots x_r(t_m) \right)^{-1} = x_r(-t_m) \cdots x_r(-t_1).
$$

Special method for unipotent elements. Returns $true$ if and only if $x$ is equal to the identity element.

Integral powers of the unipotent elements are calculated by the default methods installed in GAP. But special (more efficient) methods are installed for root elements and for the identity.

Conjugation of two unipotent elements, i.e. $x^y = y^{-1}xy$. The representation of the result will be the representation of $x$.

Special methods for unipotent elements.

$Comm$ returns the commutator of $x$ and $y$, i.e. $x^{-1} \cdot y^{-1} \cdot x \cdot y$. The second variant returns the canonical form of the commutator. In some cases it may be more efficient than $CanonicalForm( Comm( x, y ) )$.
20 ▶ IsRootElement( x )

IsRootElement returns true if and only if x is a root element, i.e. \( x = x_r(t) \) for some root \( r \). We store this property immediately after creating objects.

**Note:** the canonical form of \( x \) may be a root element even if \( x \) isn’t one.

```gap
gap> x := UnipotChevElemByRN( U_G2, [1,5,1], [2,7,-2] );
x_{1}( 2 ) * x_{5}( 7 ) * x_{1}( -2 )
gap> IsRootElement(x);
false
gap> CanonicalForm(x); IsRootElement(CanonicalForm(x));
x_{5}( 7 )
true
```

21 ▶ IsCentral( U, z )

Special method for a unipotent subgroup and a unipotent element.

## 2.4 Symbolic computation

In some cases, calculation with explicite elements is not enough. Unipot provides a way to do symbolic calculations with unipotent elements for this purpose. This is done by using indeterminates (see GAP Reference Manual, 66.1 for more information) over the underlying field instead of the field elements.

```gap
gap> U_G2 := UnipotChevSubGr("G", 2, Rationals);;
gap> a := Indeterminate( Rationals, "a" );
a
gap> b := Indeterminate( Rationals, "b", [a] );
b
gap> c := Indeterminate( Rationals, "c", [a,b] );
c
gap> x := UnipotChevElemByFC(U_G2, [ [3,1], [1,0], [0,1] ], [a,b,c] );
x_{[ 3, 1 ]}( a ) * x_{[ 1, 0 ]}( b ) * x_{[ 0, 1 ]}( c )
gap> CanonicalForm(x);
x_{[ 1, 0 ]}( b ) * x_{[ 0, 1 ]}( c ) * x_{[ 3, 1 ]}( a ) *
x_{[ 3, 2 ]}( a*c )
gap> CanonicalForm(x^-1);
x_{[ 1, 0 ]}( -b ) * x_{[ 0, 1 ]}( -c ) * x_{[ 1, 1 ]}( b*c ) *
x_{[ 2, 1 ]}( -b^2*c ) * x_{[ 3, 1 ]}( -a+b^3*c ) * x_{[ 3, 2 ]}( b^3*c^2 )
```


Index

This index covers only this manual. A page number in *italics* refers to a whole section which is devoted to the indexed subject. Keywords are sorted with case and spaces ignored, e.g., “PermutationCharacter” comes before “permutation group”.

\=, 10
\*, 11
\<, 10

C
CanonicalForm, 9
CentralElement, 6
Citing Unipot, 3
Comm, for UnipotChevElem, 11
Conjugation, of UnipotChevElem, 11

E
Elements of unipotent subgroups of Chevalley groups, 7
Equality, for UnipotChevElem, 10

G
General functionality, 4
GeneratorsOfGroup, for UnipotChevSubGr, 6

I
Inverse, for UnipotChevElem, 11
InverseOp, for UnipotChevElem, 11
IsCentral, 12
IsOne, 11
IsRootElement, 12
IsUnipotChevElem, 7
IsUnipotChevRepByFundamentalCoeffs, 7
IsUnipotChevRepByRootNumbers, 7
IsUnipotChevRepByRoots, 7
IsUnipotChevSubGr, 4

L
Less than, for UnipotChevElem, 10

M
Multiplication, for UnipotChevElem, 11

N
NegativeRootsFC, 6

O
One, for UnipotChevSubGr, 5
OneOp, for UnipotChevElem, 11
for UnipotChevSubGr, 5

P
PositiveRootsFC, 6
Powers, of UnipotChevElem, 11
PrintObj, for UnipotChevElem, 10
for UnipotChevSubGr, 5

R
Representative, 6
RootSystem, for UnipotChevSubGr, 5
Root Systems, 3

S
ShallowCopy, for UnipotChevElem, 10
Size, for UnipotChevSubGr, 5
Symbolic computation, 12

U
Unipot, 3
UNIPOT_DEFAULT_REP, 7
UnipotChevElemByFC, 8
UnipotChevElemByFundamentalCoeffs, 8
element conversion, 9
UnipotChevElemByR, 8
UnipotChevElemByRN, 7
UnipotChevElemByRootNumbers, 7
element conversion, 9
UnipotChevElemByRoots, 8
element conversion, 9
UnipotChevInfo, 4
UnipotChevSubGr, 4
Unipotent subgroups of Chevalley groups, 4

V
ViewObj, for UnipotChevElem, 10
for UnipotChevSubGr, 5