Fundamental semigroups containing a band of idempotents

El-Qallali, Fountain, Gomes and
Gould
St Andrews, September 2006

The regular case

Let S be an orthodox semigroup, that is, *S is regular and $E(S) = B$ is a band.*

S is *fundamental* if

$$\mathcal{H}^\flat = \iota.$$

Note: $S/\mathcal{H}^\flat$ is fundamental.

Meakin (71) gave a description of $\mathcal{H}^\flat$.

Since S is regular and $\mathcal{H}^\flat$ is idempotent separating, $E(S/\mathcal{H}^\flat) \cong B$:

Hall (70) constructed the largest fundamental orthodox semigroup with band of idempotents B.

Hall semigroups

- Associated with band B are two partially ordered sets $B/\mathcal{L}$ and $B/\mathcal{R}$.

- The Hall semigroup is a subsemigroup of $\mathcal{O}(B/\mathcal{L}) \times \mathcal{O}^*(B/\mathcal{R})$.

- For $e \in B$ we have

$$(\rho_e, \lambda_e) \in \mathcal{O}(B/\mathcal{L}) \times \mathcal{O}^*(B/\mathcal{R}),$$

where

$$L_b\rho_e = L_{be} = L_{ebe},\ R_b\lambda_e = R_{eb} = R_{ebe}.$$

- For $e \in B$, $\langle e \rangle = \{b \in B : b \leq e\}$.

- For $e, f \in B$ let $W_{e,f}$ denote the set of isomorphisms from $\langle e \rangle$ to $\langle f \rangle$.

- For any $e, f \in B$ and $\alpha \in W_{e,f}$ define $\alpha_\ell \in \mathcal{PO}(B/\mathcal{L}), \alpha_r^{-1} \in \mathcal{PO}^*(B/\mathcal{R})$ by

$$L_b\alpha_\ell = L_{b\alpha},\ R_b\alpha_r = R_{b\alpha^{-1}}.$$

Hall (70) *Let B be a band. Let W_B denote the set $\{(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1}) : e, f \in B, \alpha \in W_{e,f}\}$. Then W_B is a subsemigroup of*

$$\mathcal{O}(B/\mathcal{L}) \times \mathcal{O}^*(B/\mathcal{R}).$$

Further, W_B is orthodox, fundamental, and has band of idempotents

$$\overline{B} = \{(\rho_e, \lambda_e) : e \in B\} \cong B.$$

Conversely, let S be orthodox with $E(S) = B$. Then there exists a morphism $\theta : S \to W_B$ with kernel $\mathcal{H}^\flat$. Consequently, S is fundamental if and only if S is isomorphic to a full subsemigroup of W_B.

The non-regular case

Move focus from Green's relations to the '$\sim$' versions.

- Let $U \subseteq E(S)$. The relation $\widetilde{\mathcal{R}}_U$ is defined by the rule $a\,\widetilde{\mathcal{R}}_U\,b$ if and only if for all $e \in U$,

$$ea = a \Leftrightarrow eb = b.$$

- Where $U = E(S)$ we omit mention of U.
- The relation $\widetilde{\mathcal{L}}_U$ is defined dually.
- $\widetilde{\mathcal{H}}_U = \widetilde{\mathcal{R}}_U \cap \widetilde{\mathcal{L}}_U$.
- For any S and $U \subseteq E(S)$ we have

$$\mathcal{R} \subseteq \widetilde{\mathcal{R}}_U,$$

with equality if S is regular and $U = E(S)$.

Weakly U-abundant semigroups

S is *weakly U-abundant* if every $\widetilde{\mathcal{L}}_U$-class and $\widetilde{\mathcal{R}}_U$-class contains an idempotent of U.

Examples:

- Regular semigroups
- Abundant semigroups
- Ehresmann semigroups
- 2-sided C-semigroups
- Orthodox U-liberal
- 2 sided guarded semigroups

We aim to generalise Hall's approach to weakly B-abundant semigroups with B a band, satisfying (C):
$\widetilde{\mathcal{R}}_U(\widetilde{\mathcal{L}}_U)$ *is a left (right) congruence.*

The easy part

Let B be a band and let S be weakly B-abundant with (C):

The congruence $\mu_B = \widetilde{\mathcal{H}}^\flat_B$ has a description analogous to that for $\mathcal{H}^\flat$ in the orthodox case.

New definition S is B-fundamental if $\mu_B = \iota$.

- S/μ_B is B-fundamental.
- There exists a morphism

$$\theta : S \to \mathcal{O}(B/\mathcal{L}) \times \mathcal{O}^*(B/\mathcal{R})$$

with kernel μ_B.

The ***harder part*** *is to find a largest weakly B-abundant B-fundamental subsemigroup with (C) of*

$$\mathcal{O}(B/\mathcal{L}) \times \mathcal{O}^*(B/\mathcal{R}).$$

Idempotent connectedness

What makes Hall's proof work?

Let S be orthodox. Let $a \in S$ and $x \in \langle aa' \rangle$ for some $a' \in V(a)$. Then

$$xa = (aa'x)a = a(a'xa);$$

further,

$$x \mapsto a'xa$$

is an isomorphism from

$$\langle aa' \rangle \to \langle a'a \rangle.$$

The properties of (WIC) and (IC) are the analogous weak commutativity properties for weakly B-abundant semigroups with (C).

El-Qallali, Fountain, Gomes and Gould (80-06) Let B be a band. There are subsemigroups V_B, U_B and S_B of

$$\mathcal{O}(B/\mathcal{L}) \times \mathcal{O}^*(B/\mathcal{R})$$

such that

$$W_B \subseteq V_B \subseteq U_B \subseteq S_B$$

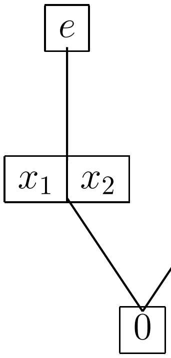
and S_B (U_B, V_B) are respectively the largest fundamental weakly B-abundant semigroups with (C) (and (WIC), (IC)).

The semigroups V_B, U_B and S_B are explicitly defined and may be constructed in case B is finite; W_B is the semigroup of regular elements of V_B.

What can you do with this?

- Weakly B-abundant semigroups with (C) and (WIC) (or (IC)) are precisely spined products of weakly $B/\mathcal{D}$-ample semigroups and U_B (or V_B). Weakly E-ample semigroups (E a semilattice) are the relevant analogue of inverse semigroups.

- We have the recipe for calculating W_B, V_B, U_B and S_B - by taking a band of small finite cardinality we can provide *finite* examples distinguishing between the classes under consideration. For example if B is:



then $V_B = W_B$ has 10 elements, but U_B has 11; consequently U_B does not have (IC).