# 1 Semigroups of languages

Example: $\{sea, river\} \cdot \{side, bed\} = \{seaside, seabed, riverside, riverbed\}$

General definition of the operation: $LM = \{uv | u \in L, v \in M\}$

This operation is associative, so we can consider semigroups of languages.

Not much is known about them - no 'normal' semigroup research at all.

A useful observation: the semigroup of languages above an alphabet $A$ is exactly the power-semigroup of the free monoid $PA^*$.

This observation is useful, but it does not allow us to use any previous research directly, because either power-semigroups of specific semigroups were studied, or power-semigroups of finite semigroups, whereas $A^*$ is infinite.

# 2 Divisors and subsemigroups

We shall study divisors and subsemigroups of semigroups of languages. The motivation comes from this observation: if $A, B$ are two languages with $A \cap B = \emptyset$ then the direct product $PA^* \times PB^*$ can be embedded into $P(A \cup B)^*$. Therefore, the class of all (finite) semigroups that are divisors of semigroups of languages is a variety (pseudovariety), and the class of all (finite) semigroups that can be embedded into semigroups of languages is a prevariety (finite quasivariety).

Therefore, we can use our usual tools for describing such classes: quasi-identities, recursive quasi-identities, or we can find an ad-hoc description.

# 3 A decomposition of the semigroup of languages

A semigroup $PA^*$ can be naturally split into a semilattice of two subsemigroups, namely, $PA^+$ and $\Lambda A^* = PA^* \setminus PA^+$. The semigroup $PA^+$ is an ideal of $PA^*$. Every cyclic subsemigroup of $PA^+$ is infinite. If we are interested in finite semigroups, we should concentrate on $\Lambda A^*$.

# 4 Positively ordered semigroups

Positively ordered semigroups featured in literature many times in different ways.

Among ordered semigroups, they can be defined by the identities $x, y \leq xy$.

Among ordered monoids, they can be defined by the identity $1 \leq x$. Such monoids are called 1-ordered monoids, and Pin used the notation $J^-$ to denote the class of finite 1-ordered monoids.

Among semigroups, positively orderable semigroups can be defined as a class generated by semigroups $\mathcal{R}_n$ or $\mathcal{U}_n$ or $\mathcal{C}_n$ from Straubing's paper or Pin's book. This is folklore.

Or (this is our result) we can describe this class by an infinite set of quasi-identities, or a finite set of recursive quasi-identities (hence, the membership problem of the class, as far as finite semigroups are concerned, is polynomial).

## 5 Divisors

Note that $\Lambda A^*$ is an ordered semigroup with respect to the inclusion order $\subseteq$.

Theorem. A semigroup is positively ordered if and only if a divisor of $\Lambda A^*$. (The divisor should be taken with respect to both the operation and the order.)

## 6 Finite subsemigroups

*This was not in the talk – I did not have enough time.*

We say that a word $u$ is a factor of a word $v$ if for some words $w_1, w_2$ we have $w_1 u w_2 = v$. Let $F(w)$ be the set of all factors of a word $w$. Let us define a semigroup $\Phi(w)$, which we shall call the semigroup of subwords of a word $w$, as the factor-semigroup $A^*/(A^* \setminus F(w))$.

Example: let us consider the word *london*. The product $on \cdot do$ is equal to *ondo*, whereas the product $on \cdot on$ is equal to zero, because there is no factor *onon* in the word *london*.

Theorem. A finite semigroup can be embedded into a semigroup of languages if and only if it can be embedded into a power-semigroup of a semigroup of subwords of a word $P\Phi(w)$.

The following corollary can be understood as a form of I. Simon's theorem.

Corollary. A finite monoid is $\mathcal{J}$-trivial if and only if it is a divisor of a power-semigroup of a semigroup of subwords of a word $P\Phi(w)$.