

# GradedRingForHomalg

## Endow Commutative Rings with an Abelian Grading

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This manual is best viewed as an HTML document. The latest version is available ONLINE at:

<http://homalg.math.rwth-aachen.de/~markus/GradedRingForHomalg/chap0.html>

An OFFLINE version should be included in the documentation subfolder of the package.

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We would like to thank the Aachen forest for being such a nice place for jogging.

# Contents

<b>1</b>	<b>Introduction</b>	<b>5</b>
1.1	What is the Role of the GradedRingForHomalg Package in the homalg Project? . . . .	5
1.2	Functionality . . . . .	5
1.3	The Math Behind This Package . . . . .	5
<b>2</b>	<b>Installation of the GradedRingForHomalg Package</b>	<b>6</b>
<b>3</b>	<b>Quick Start</b>	<b>7</b>
<b>4</b>	<b>Graded Rings</b>	<b>8</b>
4.1	Graded Rings: Category and Representations . . . . .	8
4.1.1	IsHomalgGradedRingRep . . . . .	8
4.1.2	IsHomalgGradedRingElementRep . . . . .	8
4.2	Graded Rings: Constructors . . . . .	9
4.2.1	HomalgGradedRingElement (constructor for graded ring elements using numerator and denominator) . . . . .	9
4.3	Graded Rings: Attributes . . . . .	9
4.3.1	CommonNonTrivialWeightOfIndeterminates . . . . .	9
4.3.2	WeightsOfIndeterminates . . . . .	9
4.3.3	MatrixOfWeightsOfIndeterminates . . . . .	9
4.4	Graded Rings: Operations and Functions . . . . .	9
4.4.1	UnderlyingNonGradedRing (for homalg graded rings) . . . . .	9
4.4.2	UnderlyingNonGradedRing (for homalg graded ring elements) . . . . .	10
4.4.3	Name (for homalg graded ring elements) . . . . .	10
<b>5</b>	<b>Homogeneous Matrices</b>	<b>11</b>
5.1	Homogeneous Matrices: Category and Representations . . . . .	11
5.1.1	IsHomalgMatrixOverGradedRingRep . . . . .	11
5.2	Homogeneous Matrices: Constructors . . . . .	11
5.2.1	MatrixOverGradedRing (constructor for matrices over graded rings) . . . . .	11
5.3	Homogeneous Matrices: Attributes . . . . .	12
5.3.1	DegreesOfEntries . . . . .	12
5.3.2	NonTrivialDegreePerRow . . . . .	12
5.3.3	NonTrivialDegreePerColumn . . . . .	12
5.4	Homogeneous Matrices: Operations and Functions . . . . .	12
5.4.1	UnderlyingNonGradedRing (for matrices over graded rings) . . . . .	12
5.4.2	SetMatElm (for matrices over graded rings) . . . . .	12

5.4.3	AddToMatElm (for matrices over graded rings) . . . . .	12
5.4.4	MatElmAsString (for matrices over graded rings) . . . . .	12
5.4.5	MatElm (for matrices over graded rings) . . . . .	13
<b>6</b>	<b>Examples</b>	<b>14</b>
<b>A</b>	<b>The Matrix Tool Operations</b>	<b>15</b>
A.1	The Tool Operations <i>without</i> a Fallback Method . . . . .	15
A.2	The Tool Operations with a Fallback Method . . . . .	15
A.2.1	MonomialMatrix . . . . .	15
A.2.2	RandomMatrixBetweenGradedFreeLeftModules . . . . .	16
A.2.3	RandomMatrixBetweenGradedFreeRightModules . . . . .	16
A.2.4	Diff . . . . .	16
<b>B</b>	<b>Overview of the GradedRingForHomalg Package Source Code</b>	<b>18</b>
B.1	The generic Methods . . . . .	18
B.1.1	BasisOfRowModule (for graded rings) . . . . .	18
B.1.2	DecideZeroRows (for graded rings) . . . . .	18
B.1.3	SyzygiesGeneratorsOfRows (for graded rings) . . . . .	19
B.2	Tools . . . . .	19

# Chapter 1

## Introduction

### 1.1 What is the Role of the GradedRingForHomalg Package in the homalg Project?

The homalg project [hpa10] aims at providing a general and abstract framework for homological computations. The package GradedRingForHomalg enables the homalg project to endow commutative rings in homalg with an Abelian grading.

### 1.2 Functionality

The package GradedRingForHomalg on the one hand builds on the package MatricesForHomalg and on the other hands adds functionality to MatricesForHomalg.

### 1.3 The Math Behind This Package

## Chapter 2

# Installation of the GradedRingForHomalg Package

To install this package just extract the package's archive file to the GAP `pkg` directory. `GradedRingForHomalg` also needs the package `homalg`.

By default the `GradedRingForHomalg` package is not automatically loaded by GAP when it is installed. You must load the package with

```
LoadPackage("GradedRingForHomalg");
```

before its functions become available.

Please, send me us e-mail if you have any questions, remarks, suggestions, etc. concerning this package. Also, we would be pleased to hear about applications of this package.

Mohamed Barakat and Markus Lange-Hegermann

## **Chapter 3**

### **Quick Start**

# Chapter 4

## Graded Rings

The package `GradedRingForHomalg` defines the classes of graded rings, ring elements and matrices over such rings. These three objects can be used as data structures defined in `MatricesForHomalg` on which the `homalg` project can rely to do homological computations over graded rings.

The graded rings most prominently can be used with methods known from general `homalg` rings. The methods for doing the computations are presented in the appendix (B), since they are not for external use. The new attributes and operations are documented here.

Since the objects implemented here are representations from objects elsewhere in the `homalg` project (i.e. `MatricesForHomalg`), we want to stress that there are many other operations in `MatricesForHomalg`, which can be used in connection with the ones presented here. A few of them can be found in the examples and the appendix of this documentation.

### 4.1 Graded Rings: Category and Representations

#### 4.1.1 `IsHomalgGradedRingRep`

◇ `IsHomalgGradedRingRep (R)` (Representation)

**Returns:** true or false

The representation of `homalg` graded rings.

(It is a subrepresentation of the GAP representation

`IsHomalgRingOrFinitelyPresentedModuleRep`.)

Code

```
DeclareRepresentation( "IsHomalgGradedRingRep",
  IsHomalgGradedRing and
  IsHomalgGradedRingOrGradedModuleRep,
  [ "ring" ] );
```

#### 4.1.2 `IsHomalgGradedRingElementRep`

◇ `IsHomalgGradedRingElementRep (r)` (Representation)

**Returns:** true or false

The representation of elements of `homalg` graded rings.

(It is a representation of the GAP category `IsHomalgRingElement`.)

```
Code
DeclareRepresentation( "IsHomalgGradedRingElementRep",
  IsHomalgGradedRingElement,
  [ ] );
```

## 4.2 Graded Rings: Constructors

### 4.2.1 HomalgGradedRingElement (constructor for graded ring elements using numerator and denominator)

◇ HomalgGradedRingElement(*numer*, *denom*, *R*) (function)

◇ HomalgGradedRingElement(*numer*, *R*) (function)

**Returns:** a graded ring element

Creates the graded ring element  $numer/denom$  or in the second case  $numer/1$  for the graded ring  $R$ . Both  $numer$  and  $denom$  may either be a string describing a valid global ring element or from the global ring or computation ring.

## 4.3 Graded Rings: Attributes

### 4.3.1 CommonNonTrivialWeightOfIndeterminates

◇ CommonNonTrivialWeightOfIndeterminates(*S*) (attribute)

**Returns:** a degree

The common nontrivial weight of the indeterminates of the graded ring  $S$  if it exists. Otherwise an error is issued.

### 4.3.2 WeightsOfIndeterminates

◇ WeightsOfIndeterminates(*S*) (attribute)

**Returns:** a list or listlist of integers

The list of degrees of the indeterminates of the graded ring  $S$ .

### 4.3.3 MatrixOfWeightsOfIndeterminates

◇ MatrixOfWeightsOfIndeterminates(*S*) (attribute)

**Returns:** a homalg matrix

A homalg matrix where the list (or listlist) of degrees of the indeterminates of the graded ring  $S$  is stored.

## 4.4 Graded Rings: Operations and Functions

### 4.4.1 UnderlyingNonGradedRing (for homalg graded rings)

◇ UnderlyingNonGradedRing(*R*) (operation)

**Returns:** a homalg ring

Internally there is a ring, in which computations take place.

#### 4.4.2 UnderlyingNonGradedRing (for homalg graded ring elements)

◇ UnderlyingNonGradedRing( $r$ )

(operation)

**Returns:** a homalg ring

Internally there is a ring, in which computations take place.

#### 4.4.3 Name (for homalg graded ring elements)

◇ Name( $r$ )

(operation)

**Returns:** a string

The name of the graded ring element  $r$ .

## Chapter 5

# Homogeneous Matrices

The package `GradedRingForHomalg` defines the classes of graded rings, ring elements and homogeneous matrices over such rings. These three objects can be used as data structures defined in `MatricesForHomalg` on which the `homalg` project can rely to do homological computations over graded rings.

The graded rings most prominently can be used with methods known from general `homalg` rings. The methods for doing the computations are presented in the appendix (B), since they are not for external use. The new attributes and operations are documented here.

Since the objects implemented here are representations from objects elsewhere in the `homalg` project (i.e. `MatricesForHomalg`), we want to stress that there are many other operations in `MatricesForHomalg`, which can be used in connection with the ones presented here. A few of them can be found in the examples and the appendix of this documentation.

## 5.1 Homogeneous Matrices: Category and Representations

### 5.1.1 `IsHomalgMatrixOverGradedRingRep`

◇ `IsHomalgMatrixOverGradedRingRep(A)` (Representation)

**Returns:** true or false

The representation of `homalg` matrices with entries in a `homalg` graded ring.

(It is a representation of the GAP category `IsMatrixOverGradedRing`.)

```
Code
DeclareRepresentation( "IsHomalgMatrixOverGradedRingRep",
  IsMatrixOverGradedRing,
  [ ] );
```

## 5.2 Homogeneous Matrices: Constructors

### 5.2.1 `MatrixOverGradedRing` (constructor for matrices over graded rings)

◇ `MatrixOverGradedRing(mat, S)` (function)

**Returns:** a matrix over a graded ring

Creates a matrix for the graded ring  $S$ , where  $mat$  is a matrix over `UnderlyingNonGradedRing(S)`.

## 5.3 Homogeneous Matrices: Attributes

### 5.3.1 DegreesOfEntries

◇ `DegreesOfEntries(A)` (attribute)  
**Returns:** a listlist of degrees/multi-degrees  
 The matrix of degrees of the matrix  $A$ .

### 5.3.2 NonTrivialDegreePerRow

◇ `NonTrivialDegreePerRow(A[, col_degrees])` (attribute)  
**Returns:** a list of degrees/multi-degrees  
 The list of first nontrivial degree per row of the matrix  $A$ .

### 5.3.3 NonTrivialDegreePerColumn

◇ `NonTrivialDegreePerColumn(A[, row_degrees])` (attribute)  
**Returns:** a list of degrees/multi-degrees  
 The list of first nontrivial degree per column of the matrix  $A$ .

## 5.4 Homogeneous Matrices: Operations and Functions

### 5.4.1 UnderlyingNonGradedRing (for matrices over graded rings)

◇ `UnderlyingNonGradedRing(mat)` (operation)  
**Returns:** a homalg ring  
 The nongraded ring underlying `HomalgRing(mat)`.

### 5.4.2 SetMatElm (for matrices over graded rings)

◇ `SetMatElm(mat, i, j, r, R)` (operation)

Changes the entry  $(i, j)$  of the matrix  $mat$  to the value  $r$ . Here  $R$  is the graded homalg ring involved in these computations.

### 5.4.3 AddToMatElm (for matrices over graded rings)

◇ `AddToMatElm(mat, i, j, r, R)` (operation)

Changes the entry  $(i, j)$  of the matrix  $mat$  by adding the value  $r$  to it. Here  $R$  is the (graded) homalg ring involved in these computations.

### 5.4.4 MatElmAsString (for matrices over graded rings)

◇ `MatElmAsString(mat, i, j, R)` (operation)  
**Returns:** a string  
 Returns the entry  $(i, j)$  of the matrix  $mat$  as a string. Here  $R$  is the (graded) homalg ring involved in these computations.

### 5.4.5 MatElm (for matrices over graded rings)

◇ `MatElm(mat, i, j, R)`

(operation)

**Returns:** a graded ring element

Returns the entry  $(i, j)$  of the matrix  $mat$ . Here  $R$  is the (graded) homalg ring involved in these computations.

## **Chapter 6**

### **Examples**

# Appendix A

## The Matrix Tool Operations

The functions listed below are components of the `homalgTable` object stored in the ring. They are only indirectly accessible through standard methods that invoke them.

### A.1 The Tool Operations *without* a Fallback Method

There are matrix methods for which `homalg` needs a `homalgTable` entry for non-internal rings, as it cannot provide a suitable fallback. Below is the list of these `homalgTable` entries.

### A.2 The Tool Operations with a Fallback Method

These are the methods for which it is recommended for performance reasons to have a `homalgTable` entry for non-internal rings. `homalg` only provides a generic fallback method.

#### A.2.1 MonomialMatrix

◇ `MonomialMatrix( $d$ ,  $R$ )` (operation)

**Returns:** a `homalg` matrix

The column matrix of  $d$ -th monomials of the `homalg` graded ring  $R$ .

Example

```
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> S := GradedRing( R );;
gap> m := MonomialMatrix( 2, S );
<A ? x 1 matrix over a graded ring>
gap> NrRows( m );
6
gap> m;
<A 6 x 1 matrix over a graded ring>
gap> Display( m );
z^2,
y*z,
y^2,
x*z,
x*y,
x^2
```

### A.2.2 RandomMatrixBetweenGradedFreeLeftModules

◇ `RandomMatrixBetweenGradedFreeLeftModules(degreesS, degreesT, R)` (operation)

**Returns:** a homalg matrix

A random  $r \times c$ -matrix between the graded free *left* modules  $R^{(-degreesS)} \rightarrow R^{(-degreesT)}$ , where  $r = \text{Length}(degreesS)$  and  $c = \text{Length}(degreesT)$ .

Example

```
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "a,b,c";;
gap> S := GradedRing( R );;
gap> rand := RandomMatrixBetweenGradedFreeLeftModules( [ 2, 3, 4 ], [ 1, 2 ], S );
<A 3 x 2 matrix over a graded ring>
gap> Display( rand );
a-2*b+2*c,                2,
a^2-a*b+b^2-2*b*c+5*c^2,  3*c,
2*a^3-3*a^2*b+2*a*b^2+3*a^2*c+a*b*c-2*b^2*c-3*b*c^2-2*c^3,a^2-4*a*b-3*a*c-c^2
```

### A.2.3 RandomMatrixBetweenGradedFreeRightModules

◇ `RandomMatrixBetweenGradedFreeRightModules(degreesT, degreesS, R)` (operation)

**Returns:** a homalg matrix

A random  $r \times c$ -matrix between the graded free *right* modules  $R^{(-degreesS)} \rightarrow R^{(-degreesT)}$ , where  $r = \text{Length}(degreesT)$  and  $c = \text{Length}(degreesS)$ .

Example

```
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "a,b,c";;
gap> S := GradedRing( R );;
gap> rand := RandomMatrixBetweenGradedFreeRightModules( [ 1, 2 ], [ 2, 3, 4 ], S );
<A 2 x 3 matrix over a graded ring>
gap> Display( rand );
a-2*b-c,a*b+b^2-b*c,2*a^3-a*b^2-4*b^3+4*a^2*c-3*a*b*c-b^2*c+a*c^2+5*b*c^2-2*c^3,
-5,      -2*a+c,      -2*a^2-a*b-2*b^2-3*a*c
```

### A.2.4 Diff

◇ `Diff(D, N)` (operation)

**Returns:** a homalg matrix

If  $D$  is a  $f \times p$ -matrix and  $N$  is a  $g \times q$ -matrix then  $H = \text{Diff}(D, N)$  is an  $fg \times pq$ -matrix whose entry  $H[g*(i-1)+j, q*(k-1)+l]$  is the result of differentiating  $N[j, l]$  by the differential operator corresponding to  $D[i, k]$ . (Here we follow the Macaulay2 convention.)

Example

```
gap> S := HomalgFieldOfRationalsInDefaultCAS( ) * "a,b,c" * "x,y,z";;
gap> D := HomalgMatrix( "[ \
> x,2*y, \
> y,a-b^2, \
> z,y-b \
> ]", 3, 2, S );;
<A 3 x 2 matrix over an external ring>
gap> N := HomalgMatrix( "[ \
> x^2-a*y^3,x^3-z^2*y,x*y-b,x*z-c, \
> x,      x*y,      a-b,  x*a*b \
> ]", 2, 4, S );;
```

```
<A 2 by 4 matrix over an external ring>
gap> H := Diff( D, N );
<A 6 x 8 matrix over an external ring>
gap> Display( H );
2*x,      3*x^2, y, z, -6*a*y^2, -2*z^2, 2*x, 0,
1,        y,      0, a*b, 0,      2*x,   0,  0,
-3*a*y^2, -z^2,  x, 0, -y^3,    0,    0,  0,
0,        x,      0, 0,  0,      0,    1,  b*x,
0,        -2*y*z, 0, x, -3*a*y^2, -z^2,  x+1, 0,
0,        0,      0, 0,  0,      x,    1, -a*x
```

## Appendix B

# Overview of the GradedRingForHomalg Package Source Code

This appendix is included in the documentation to shine some light on the mathematical backgrounds of this Package. Neither is it needed to work with this package nor should the methods presented here be called directly. The functions documented here are entries of the so called ring table and not to be called directly. There are higher level methods in declared and installed in MatricesForHomalg, which call this functions ( $\rightarrow$  ?MatricesForHomalg:The Basic Matrix Operations).

### B.1 The generic Methods

We will present some methods as an example, to show the idea:

#### B.1.1 BasisOfRowModule (for graded rings)

◇ BasisOfRowModule(*M*) (function)

**Returns:** a distinguished basis (i.e. a distinguished generating set) of the module generated by *M*

```
Code
BasisOfRowModule :=
function( M )
  return MatrixOverGradedRing(
    BasisOfRowModule( UnderlyingMatrixOverNonGradedRing( M ) ),
    HomalgRing( M ) );
end,
```

#### B.1.2 DecideZeroRows (for graded rings)

◇ DecideZeroRows(*A*, *B*) (function)

**Returns:** a reduced form of *A* with respect to *B*

```
Code
DecideZeroRows :=
function( A, B )
  return MatrixOverGradedRing(
    DecideZeroRows( UnderlyingMatrixOverNonGradedRing( A ) ),
```

```

        UnderlyingMatrixOverNonGradedRing( B ) ),
    HomalgRing( A ) );
end,

```

### B.1.3 SyzygiesGeneratorsOfRows (for graded rings)

◇ SyzygiesGeneratorsOfRows (M)

(function)

**Returns:** a distinguished basis of the syzygies of the argument

```

Code
SyzygiesGeneratorsOfRows :=
function( M )
    return MatrixOverGradedRing(
        SyzygiesGeneratorsOfRows( UnderlyingMatrixOverNonGradedRing( M ) ),
        HomalgRing( M ) );
end,

```

## B.2 Tools

The package GradedRingForHomalg also implements tool functions. These are referred to from MatricesForHomalg automatically. We list the implemented methods here and refer to the MatricesForHomalg documentation ([→ ?MatricesForHomalg: The Matrix Tool Operations and ?MatricesForHomalg:RingElement](#)) for details. All tools functions from MatricesForHomalg not listed here are also supported by fallback tools.

- IsZero
- IsOne
- Minus
- DivideByUnit
- IsUnit
- Sum
- Product
- ShallowCopy
- ZeroMatrix
- IdentityMatrix
- AreEqualMatrices
- Involution
- CertainRows
- CertainColumns

- UnionOfRows
- UnionOfColumns
- DiagMat
- KroneckerMat
- MulMat
- AddMat
- SubMat
- Compose
- NrRows
- NrColumns
- IsZeroMatrix
- IsDiagonalMatrix
- ZeroRows
- ZeroColumns

# References

- [hpa10] T. homalg project authors. *The homalg project*, 2003-2010. <http://homalg.math.rwth-aachen.de/>. 5

# Index

`GradedRingForHomalg`, 5

`AddToMatElm`  
for matrices over graded rings, 12

`BasisOfRowModule`  
for graded rings, 18

`CommonNonTrivialWeightOfIndeterminates`, 9

`DecideZeroRows`  
for graded rings, 18

`DegreesOfEntries`, 12

`Diff`, 16

`HomalgGradedRingElement`  
constructor for graded ring elements using a given numerator and one as denominator, 9  
constructor for graded ring elements using numerator and denominator, 9

`IsHomalgGradedRingElementRep`, 8

`IsHomalgGradedRingRep`, 8

`IsHomalgMatrixOverGradedRingRep`, 11

`MatElm`  
for matrices over graded rings, 13

`MatElmAsString`  
for matrices over graded rings, 12

`MatrixOfWeightsOfIndeterminates`, 9

`MatrixOverGradedRing`  
constructor for matrices over graded rings, 11

`MonomialMatrix`, 15

`Name`  
for homalg graded ring elements, 10

`NonTrivialDegreePerColumn`, 12

`NonTrivialDegreePerRow`, 12

`RandomMatrixBetweenGradedFreeLeftModules`, 16

`RandomMatrixBetweenGradedFreeRightModules`, 16

`SetMatElm`  
for matrices over graded rings, 12

`SyzygiesGeneratorsOfRows`  
for graded rings, 19

`UnderlyingNonGradedRing`  
for homalg graded ring elements, 10  
for homalg graded rings, 9  
for matrices over graded rings, 12

`WeightsOfIndeterminates`, 9