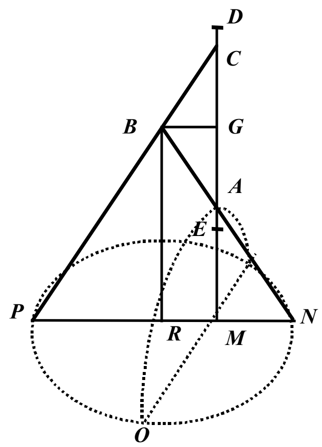
§22.1. ***Synopsis: Chapter Twenty Two.***

The theorem stated below is solved by making use of the properties of the regular hyperbola, treated as a conic section, as described in the notes in section 3.

§22.2. *Chapter Twenty Two.* [p.60.]

Given the base, the difference of the lengths of the legs, & the area of a triangle: to find the legs [i.e. sides].

Let the base be DE and the difference of the legs CA [Fig. 22-1]: and let GD equal GE, GC equal GA. GB drawn perpendicular to the line DE is the mean proportional between EC & CD[1]. And the lines CBP, BAN are drawn; and BR parallel to DE. Then the area is divided by half the base DE; the quotient is the altitude of the triangle OM. & if RM is equal to the line BG & the angle RMO is right; RO is the radius of the circle PON. And if following the length PR of the radius found, the parallel line BR is drawn, intersecting the line CB in the point P. & the line DE is drawn perpendicular to PRMN, & let OM in the plane be projected perpendicular to the diameter PN of the circle from the point M, & equal to the altitude found: the difference of the lines DO, EO is equal to the line AC given, by Prop.51, Book 3, *Apollonius of Perga*. & DEO is the triangle of which the sides are to be found[2].



[Figure 22-1]

The idea behind this is more apparent with numbers. Let the base be DE, 25; CA, 20; Area 225; DC is 2_. CE 22_. The product of the sides of the rectangle DCE 56_, of which the root BG is 7_, is the mean proportional between DC & CE.

[p.61.]

			Logarithms
pro--	DC	2_	0,39794,001
port	BG	7_	0,87506,126
-ions	CE	22_	1,35218,252

[Table 22-1]

The given area 225, divided by the half of DE 12_, gives the quotient OM 18 , the altitude of the triangle, of which the square 324, added to the rectangle DCE 56_ [= BG^2] gives 380_ , the square of the radius of the circular base PON of the cone, of which the root 19_ is the radius PR. Also: BG,7_; GC,10; PR,19_ ; RB,26 are in proportion.

				Logarithms
pro-	BG	7_	Compl. Arith.	9,12493,874
port-	GC	10	- - - - - - - - - - - - -	1,00000,000
ion-	PR	19_	- - - - - - - - - - - - -	1,29003,461
-als.	RB	26	- - - - - - - - - - - -	(1)1,41497,335

[Table 22-2]

And GM equals BR. But of the lines GE, GM, the difference is EM, 13_ : yet the sum is DM, 38_. The squares of which 1482_, 182_ added in turn to the square of the altitude OM 324, gives the squares of the lines DO, EO 1806_, 506_ . The lines are DO, 42_; EO, 22_; of which the difference is equal to the line CA, 20.

It will not be troublesome to add another example where BG is irrational.

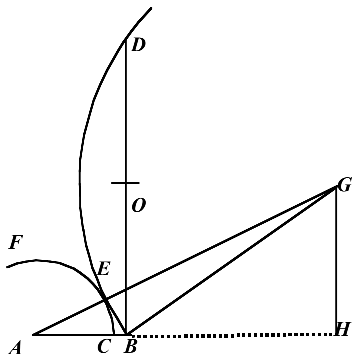
Let DE be 26; & CA, 22. The area 156. DC is 2, CE 24, & BG, _.48, with altitude OM 12, of which the square 144 added to the square of BG, 48, gives the square of the radius RO, or RP _.192. & with BG, GC : PR, RB proportional; they are also proportional as squares.

				Logarithms
pro-	Sq. BG	48	Comp. Arith.	8,31875,876
port-	Sq. GC	121		2,08278,537
ion-	Sq. PR	192		2,28330,123
als	Sq. RB	484		(1)2,68484,536

[Table 22-3]

And RB itself or GM is 22. DM is 35, & EM 9, the squares of which 1225, & 81, added [in turn] to the square of the other altitude OM 144, make 1369 and 225, the squares of the lines 37 & 15 required. The difference of which is equal to the line CA, 22.

François Viète, at the end of his *Apollonius Gallus,* explains this matter otherwise, in this manner. Let the base AB be given; the difference of the sides sought be AC; the altitude of the given



[Figure 22-2].

triangle BO, & let DB be perpendicular to the base, & double the given altitude BO: & with centre A and radius AC the arc CEF drawn. Then, by problem 8 (of *Apollonius Gallus)*, another arc BED is drawn touching the first arc at the point E, & crossing through the points B & D, the vertex G of the triangle is the centre of this arc, of which the side AG is equal to the sides BG, & of the given difference AC.

The first method seems to be superior in this way, since the whole business has been understood in the volume situation [i.e. in 3 dimensions]; but here in the same plane location, this second method is nevertheless not less laborous, but easier to understand than the other. And if we again wish to judge this second result with numbers: in that first case, all is easier to explain, here truly not without a great deal of trouble and difficulty[3].

§22.3. *Notes on Chapter 22.*

1 We are dealing with the two branches of a regular hyperbola, of which the lower branch is drawn as a dotted curve as a vertical section of the right cone through the point B, passing through O and the vertex A. E & D are the foci of the hyperbola, and C is the other vertex. Hence, the difference of the lengths OD and OE is always the distance between the vertices AC, according to the definition of the hyperbola. The altitude OM is drawn from some point on the axis to give the required numerical area for the triangle DEO.

2 From similar triangles, we have BG/PM = CG/CM, and BG/MN = AG/AM; also PM.MN = OM^2. Hence, BG^2 = (CG/CM).(AG/AM).OM^2. To ease matters, we continue now using modern analytic notation, where we write the equation of the hyperbola in the vertical plane OEM in the form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,with eccentricity e > 1. In this case, with origin G, we have designated GM as the x ordinate, and OM as the y co-ordinate (for OM is a line parallel to the y-axis); and setting |GC| = GA = a. Consequently: $BG^2 = \frac{a}{(x+a)}\frac{a}{(x-a)}.y^2 = \frac{a^2y^2}{(x^2-a^2)} = b^2$. The mean proportional

quoted by Briggs: $BG^2 = EC.CD$ now reduces to the well-known result $b^2 = a^2(e^2 - 1)$; note that the foci are now the points E (ae, 0) and C (-ae, 0). The identification of the 'minor' diameter b for the hyperbola written in standard form for a right cone, in analogy with the similar result for the ellipse, as being the perpendicular distance of the plane containing the curve from the vertex of the cone, is one not normally found in elementary geometry texts, which are concerned with properties in the plane of the curve only. For this reason Briggs' work is of extra interest here, as he makes this useful observation possible. Also, tan $\alpha = b/a$, where α is the half angle formed by the vertex of the cone.

[3] Viète's method would certainly appear to be the more straightforward and easily executed as a drawing. However, Briggs's concerns lay with the calculation of the position of G.

§22.4. **Caput XXII**. [p.60.]

Datis Basi, differentia crurum & Area trianguli: invenire crura.

Fiat DE basis & CA differentia crurum: & sint GD, GE: GC, GA aequales: & GB perpendicularis rectae DE fiat media proportionalis inter EC & CD. & ducantur rectae CBP, BAN : & BR parallela rectae DE. Deinde divisa Area data per semissem basis DE; Quotus erit Altitudo trianguli OM. & si RM aequetur rectae BG & angulus RMO sit rectus; erit RO radius circuli PON. Et si secundum distantiam radij inventi PR , ducatur parallela rectae BR, intersecans rectam CB in puncto P. & ducatur PRMN perpendicularis rectae DE, & sit OM in plano subiecto perpendicularis diametro circuli PN in puncto M, & aequalis inventae altitudini:erit differentia rectarum DO, EO aequalis rectae AC datae, per 51.p.3.lib. Apollonij Pergaei. & triangulum quaesitorum laterum erit DEO.

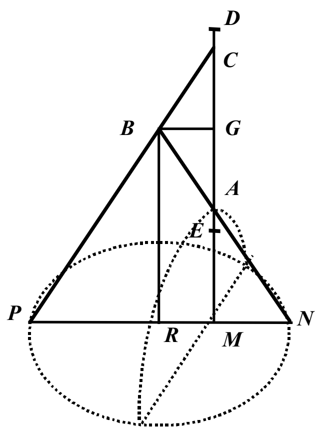
In numeres res erit manifestior. Sit data basis DE, 25. CA, 20, Area 225. Erit DC 2_. CE 22_. factus seu rectangulum DCE 56_, cuius latus est BG 7_ media proportionalis inter DC & CE.

[p.61.]

			Logarithmi.
prop.	DC	2_	0,39794,001
	BG	7_	0,87506,126
	CE	22_	1,35218,252

Area data 225, divisa per semissem DE 12_, dat quotum OM 18 , altitudinem trianguli, cuius quadratum 324, adiectum oblongo DCE 56_ , dat 380_ , quadratum radij conicae basis circularis PON, cuius latus 19_ est radius PR. Sunt autem: BG,7_; GC,10; PR,19_ ; RB,26 proportionales.

				Logarithmi.
pro-port-ion--als.	BG	7_	Compl. Arith.	9,12493,874
	GC	10	- - - - - - - - - - - - -	1,00000,000
	PR	19_	- - - - - - - - - - - - -	1,29003,461
	RB	26	- - - - - - - - - - - -	(1)1,41497,335

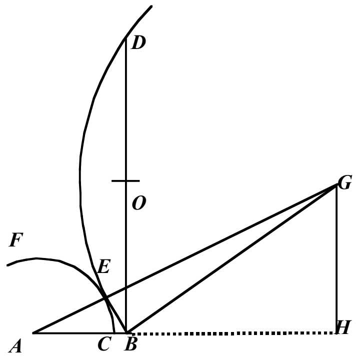


Et GM aequalis rectae BR. rectarum autem GE, GM, differentia est EM, 13_ : summa vero est DM, 38_. quarum quadrata 1482_, 182_ adiecta quadrato rectae OM altitudinis 324, dant quadrata rectarum DO, EO 1806_, 506_ . & sunt rectae DO, 42_; EO, 22_; quarum differentia est aequalis rectae CA, 20.

Aliud exemplum non pigebit adijcere ubi BG est irrationalis.

Sit DE 26; CA, 22. area 156. Erunt DC 2, CE 24, & BG, _.48, altitudo OM 12, cuius quadratum 144 additum quadrato rectae BG, 48, dat quadratum radij RO, vel RP _.192. & cum BG, GC : PR, RB sint proportionales;erunt earum quadrata etiam proportionalia.

			Logarithmi.
Sq. BG	48	Comp. Arith.	8,31875,876
Sq. GC	121		2,08278,537
Sq. PR	192		2,28330,123
Sq. RB	484		(1)2,68484,536



Et ipsa RB vel GM erit 22. DM vero erit 35, & EM 9, quarum quadrata 1225, & 81, adiecto utrique quadrato altitudinis OM 144, faciunt1369 & 225, quadrata laterum quaesitorum 37 & 15. quorum differentia aequatur rectae CA, 22.

Fr. Vieta, ad finem sui Apollonij Galli, hoc negotium expedit aliter, ad hunc modum. Esto data basis AB; differentia laterum quaesitorum AC; altitudo trianguli data BO, & fiat DB perpendicularis basi, & dupla datae altitudinis BO: & centro A radio AC describatur peripheria CEF. Deinde per Prob.8. Apoll. Galli, describatur alia peripheria BED contingens priorem peripheriam in puncto E, & transiens per puncta B D, erit centrum huius peripheriae G vertex trianguli , cuius latus AG aequatur lateri BG, & datae differentiae AC.

Atque hic modus priorem in hoc superare videtur, quod illic omnia fieri intelliguntur in loco solido, hic autem in eodem loco plan. est tamen hic modus non minus operosus, aut magis certus, quam ille alter. & si ista ad numeros revocare velimus, illic omnia facillime expediuntur; hic vero non sine magna & difficili molestia..