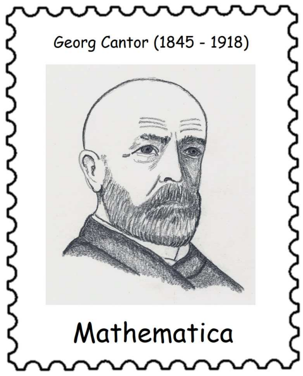
Georg Cantor (March 3, 1845 – January 6, 1918)

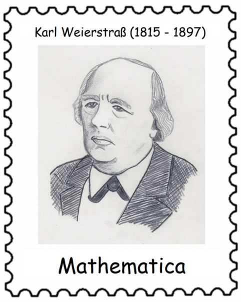
by Heinz Klaus Strick, Germany

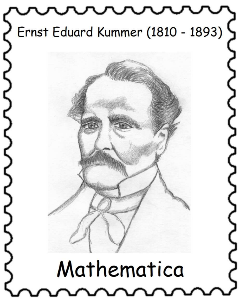


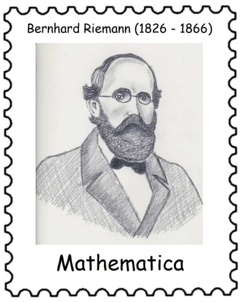
There is hardly another mathematician whose reputation among his contemporary colleagues reflected such a wide disparity of opinion: for some, Georg Ferdinand Ludwig Philipp Cantor was a corruptor of youth (Kronecker), while for others, he was an exceptionally gifted mathematical researcher (David Hilbert 1925: *Let no one be allowed to drive us from the paradise that Cantor created for us*.)

Georg Cantor's father was a successful merchant and stockbroker in St. Petersburg, where he lived with his family, which included six children, in the large German colony until he was forced by ill health to move to the milder climate of Germany. In Russia, Georg was instructed by private tutors. He then attended secondary schools in Wiesbaden and Darmstadt. After he had completed his schooling with excellent grades, particularly in mathematics, his father acceded to his son's request to pursue mathematical studies in Zurich. Georg Cantor could equally well have chosen a career as a violinist, in which case he would have continued the tradition of his two grandmothers, both of whom were active as respected professional musicians in St. Petersburg.

When in 1863 his father died, Cantor transferred to Berlin, where he attended lectures by Karl Weierstrass, Ernst Eduard Kummer, and Leopold Kronecker. On completing his doctorate in 1867 with a dissertation on a topic in number theory, Cantor did not obtain a permanent academic position. He taught for a while at a girls' school and at an institution for training teachers, all the while working on his habilitation thesis, which led to a teaching position at the university in Halle.

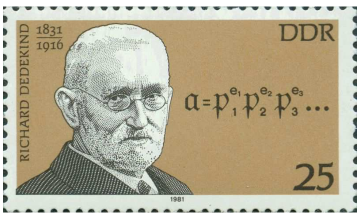






(drawings: © Andreas Strick)

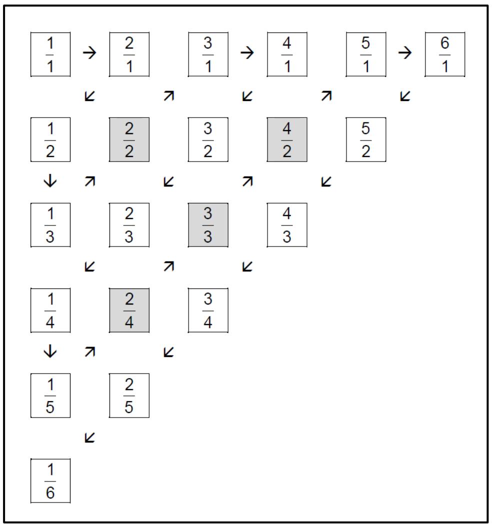
Encouraged by his Halle colleague Eduard Heine, he sought the solution to a problem on which Heine himself, and also Dirichlet, Lipschitz, and Riemann, had worked in vain: his proof of the uniqueness of representation of functions by trigonometric series (Fourier series) led to his appointment as an assistant professor at the University of Halle. In investigating on which sets of numbers the uniqueness theory held, he realized that there was as yet no precise definition of the set of real numbers.



On this topic he developed with Richard Dedekind, whom he had met by chance during a vacation in Switzerland, an active and fruitful correspondence that became the basis for Dedekind's definition of the irrational numbers through so-called Dedekind cuts and Cantor's development of set theory (originally called by him the theory of aggregates).

The problem of how one might compare the numbers of points on two segments of different lengths goes back to GALILEO: both segments contain infinitely many points, but is the “infinity” of one segment greater than the “infinity” of the other? And can one say that there are fewer perfect squares than natural numbers, since not all natural numbers are squares?

The univocal—in the case of finite sets—notion of number of elements is simply not applicable to infinite sets. So at this point, CANTOR introduced the notion of cardinality. Toward the end of 1873, he discovered how to associate a unique positive natural number with every rational number. In this “enumeration” of the rational numbers, one deletes all the fractions shaded in grey in the figure, since these have already been accounted for (those whose numerator and denominator are not relatively prime). The method is called CANTOR’s *first diagonalization procedure*. If one lists alternately the positive fractions shown in the figure and their negatives, then in this way, one encompasses all the rational numbers. It is thereby shown that the set of natural numbers and the set of rational numbers have the same cardinality:



$|\mathbb{N}| = |\mathbb{Q}|$.

CANTOR then developed the idea further, proving the countability (i.e., having the same cardinality as that of the natural numbers) of the set of algebraic numbers, which is the set of solutions of polynomial equations with integer coefficients.

He worked in vain, using a similar procedure, to prove the countability of the real numbers. But then only a few weeks later, in December 1873, he was able to prove, by an indirect argument, the negation of his original conjecture. In his proof, he restricted attention to the set of real numbers between 0 and 1:

$x_1 = 0{,}a_{11}\,a_{12}\,a_{13}\,a_{14}\,\ldots,$
$x_2 = 0{,}a_{21}\,a_{22}\,a_{23}\,a_{24}\,\ldots,$
$x_3 = 0{,}a_{31}\,a_{32}\,a_{33}\,a_{34}\,\ldots,$
$x_4 = 0{,}a_{41}\,a_{42}\,a_{43}\,a_{44}\,\ldots,$

Assume that the set of real numbers in the given interval is countable. Then there exists a sequence $x_1, x_2, x_3, x_4, \ldots$ that includes all the real numbers in the interval. However, that cannot possibly be the case, since the sequence does not contain the real numbers that differ in the first digit of their decimal expansion from x_1 and in their second digit from x_2 and in their third digit from x_3, and so on; furthermore, not all the digits can be equal to 9 (CANTOR’s *second diagonalization procedure*).

CANTOR realized that there must be at least two different types of “infinity,” and he introduced the use of the Hebrew letter aleph to denote the cardinalities of the two sets:

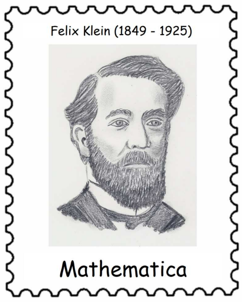
$\aleph_0 = |\mathbb{N}| = |\mathbb{Q}| < \aleph_1 = |\mathbb{R}|$.

Since the set of real numbers turned out not to uncountable (today we call such a set uncountably infinite or simply uncountable), whereas the subset of the real numbers comprising all algebraic numbers is countable, it follows that the set of transcendental numbers is also uncountable, or to put it somewhat sloppily, that “almost all” real numbers are transcendental.

In 1874, Cantor, who at first considered it superfluous, found time to use in his search a method by means of which, to his great surprise, one can establish a one-to-one correspondence between the points of the unit square and those of the unit interval [0, 1]: to a point (x | y) with $x = 0.a_1a_2a_3\ldots$ and $y=0.b_1b_2b_3\ldots$ is assigned the number $z=a_1b_1a_2b_2a_3b_3\ldots$ of the interval [0, 1]. (The inverse of this mapping is not quite one-to-one, since, for example, one has 0.5 = 0.49999…; however, Cantor found a way around this imperfection.)

Dedekind, who was the first to hear of these sensational results (Cantor: *Je le vois, mais je ne le crois pas*!), was surprised that a two-dimensional surface should have the same cardinality as a one-dimensional interval. However, he saw no flaw in Cantor’s reasoning.

The year 1874 also saw Cantor’s marriage (to a friend of his sister); their wedding journey took them to Interlaken, where Cantor met often with Dedekind.



When in 1877, Cantor submitted a lengthy article on his researches to Crelle’s Journal (including a proof that the unit interval has the same cardinality as the *n*-dimensional unit cube), opposition arose for the first time. Kronecker (*God created the integers; all else is the work of man*) tried to prevent publication. It was printed only after the intervention of Dedekind. Cantor submitted no further articles to Crelle’s Journal; his next six articles appeared in the *Mathematische Annalen*, supported by Felix Klein.

In 1879, Cantor finally was promoted to a full professorship in Halle, following a long but vain hope of being called to the more prestigious University of Berlin. When in 1881, Heine died, Cantor hoped to bring Dedekind to Halle to occupy the now open chair as Heine’s successor. But when Dedekind declined the offer, the mutually fruitful correspondence between Cantor and Dedekind came to an end.

Cantor’s publications in the *Mathematische Annalen*, including *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* (Foundations of a general theory of aggregates) did not receive the hoped-for positive response. Indeed, the number of opponents only increased.

In 1874, those around Cantor began for the first time to notice that he was suffering from a deep depression. This was aggravated by the fact that after a number of promising results, Cantor was unable either to prove or disprove the so-called continuum hypothesis: there is no set whose cardinality lies between that of the rational numbers and that of the real numbers. (It was not until the 1960s that this problem was solved when Paul Cohen showed that the statement of the continuum hypothesis cannot be derived from the axioms of set theory.)

Cantor wrote further articles, which he submitted to the *Acta Mathematica*, in Sweden, including an article about what is today called the Cantor set, the first fractal in the history of mathematics (*De la puissance des ensembles parfait de points*; *Über die Mächtigkeit von perfekten Punktmengen*; On the cardinality of perfect sets of points):

Beginning with a line segment, namely the interval [0, 1], the middle thirds of the intervals that arise are successively removed. All the sets of points in the resulting sequence contain infinitely many points, yet the sum of lengths of the individual intervals approaches zero.

The publisher of the journal, Magnus Gösta Mittag-Leffler, asked Cantor to withdraw one of his articles from consideration, asserting that it was a century too soon for its publication. Cantor agreed to this proposal—offered by Mittag-Leffler in a spirit of friendship – but he then ended contact with him.

For a while, Cantor shifted the focus of his activity: He purchased a suitably large house into which he moved with his wife and six children. He also worked on founding the *German Mathematical Society*, which held its first meeting in Halle in 1891. As a gesture of reconciliation, he invited Kronecker to give the opening address, which Kronecker had to decline following an accident involving his wife. Cantor was chosen president of the society for a three-year term, but he was unable to attend the society's general assembly in 1893, in Munich, on account of illness.

In 1895 and 1897 there appeared comprehensive presentations of his set theory, containing the definition of a set that is familiar to all mathematicians today: A set is a gathering together into a whole of definite, distinct objects of our perception and of our thought—which are called elements of the set.

Finally, at a mathematical conference in Zurich in 1897, Cantor's achievements were laid out in lectures by Adolf Hurwitz and Jacques Hadamard; there was also a reconciliation with Dedekind. However, Cantor never regained his earlier creativity. In the meantime, he had discovered paradoxes associated with the definitions that he had made, and he could discover no way of resolving those difficulties.

For Henri Poincaré, the paradoxes were proof that set theory represented a false path: The greater portion of the ideas of Cantor should be banished from mathematics once and for all.

Cantor fled into another world: with great obsession, he attempted to prove that it was in fact Francis Bacon who had created the works of Shakespeare.

His health underwent wild fluctuations. He was able to fulfil his university obligations only at times. At the International Congress of Mathematicians in 1904, in Heidelberg, a speaker maintained in his presence that there were substantial errors in the foundations of Cantor's set theory. Although Ernst Zermelo was able to defang the accusations immediately, Cantor was deeply wounded.

In 1911, he was invited as a guest of honour to St. Andrew's University, in Scotland, on the occasion of the 500th anniversary of the founding of the university. There, he hoped to meet Bertrand Russell, who in his recently published *Principia Mathematica* had dealt with Cantor's set theory and its paradoxes. When Russell was unable to attend on account of illness, Cantor engaged in discussions only on the subject of his Bacon–Shakespeare theory.

Cantor retired in 1913. A planned celebration on the occasion of his seventieth birthday was cancelled due to the outbreak of World War I. His health worsened, in part due to a lack of adequate nourishment during the war. He spent his last year in a psychiatric clinic in Halle.

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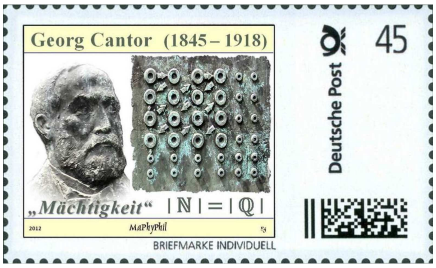

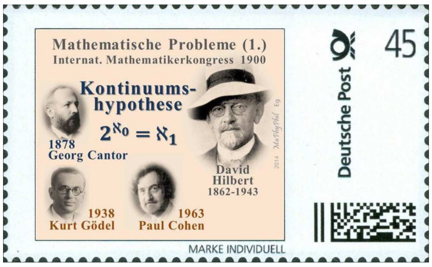

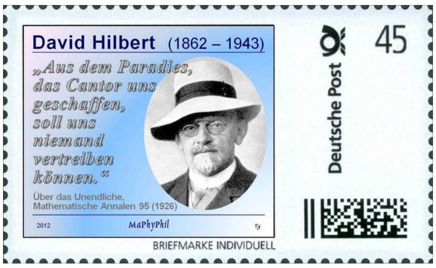

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