**PAOLO RUFFINI** (September 22, 1765 – May 10, 1822)

by HEINZ KLAUS STRICK, Germany


(drawings © Andreas Strick)

Born in the small town of Valentano, halfway between Rome and Siena, PAOLO RUFFINI grew up as the son of a doctor. Later the family moved to Modena (Emilia-Romagna), where PAOLO began studying mathematics, medicine, philosophy and literature at the university at the age of 18.

When his calculus professor was appointed administrator of the princely estates by the Duke of Modena and Reggio, the student RUFFINI was assigned to give the calculus lectures. In the following year (1788), he finished his studies with degrees in philosophy, medicine and mathematics and was appointed Professor of the Fundamentals of Calculus.

And when three years later his former geometry professor had to give up teaching for health reasons, RUFFINI's appointment as Professor of the Foundations of Mathematics followed. In the same year he was also licensed to practise medicine.


In 1795, the French revolutionary troops began the conquest of northern Italy, and finally, in June 1797, NAPOLEON proclaimed the *Cisalpine Republic* as a French "daughter republic" with Milan as its capital. Against his will, RUFFINI was appointed as a delegate to one of the chambers of the newly founded state. He soon succeeded in relinquishing this unloved office and resumed his university activities. This was short-lived, however, as he was obliged to take an oath to the constitution of the republic, which he refused to do for religious reasons.

RUFFINI bore his dismissal from the university service with great equanimity, because it gave him more time to treat his patients, to whom he devoted himself with great dedication. In addition, he now had the leisure to deal with a mathematical problem that has occupied him for years: The question of the solvability of $5^{\text{th}}$ degree equations by radicals.

The Babylonians already knew methods for solving quadratic equations and these were systematised around 830 by AL-KHWARIZMI. In the course of the 16th century, SCIPIONE DEL FERRO, GIROLAMO CARDANO, NICCOLÒ TARTAGLIA, LODOVICO FERRARI and finally FRANÇOIS VIÈTE developed methods to solve arbitrary equations of the $3^{\text{rd}}$ or $4^{\text{th}}$ degree.






Since at first only positive coefficients were allowed, one had to make several case distinctions. CARDANO showed that this is not necessary, but that for the solution of the reduced equation of $3^{\text{rd}}$ degree $z^3 + pz + q = 0$ only different cases have to be considered with regard to the *discriminant* $\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$.



Through the use of complex numbers by RAFAEL BOMBELLI, it was finally clear that equations up to degree 4 are solvable by radicals, i.e. that the solutions could be represented with the help of formulas in which only addition, subtraction, multiplication and division as well as root extraction occured as operations.

Finding a corresponding solution formula for equations of the 5th degree seemed to be only a matter of time. But although many mathematicians, including LEONHARD EULER, tried to find such a method of solution, the search was in vain.

In 1770, JOSEPH LOUIS LAGRANGE published the treatise *Réflexions sur la résolution algébrique des équations*, in which he investigated so-called *resolvents*, i.e. auxiliary quantities that play a role in the solution procedure (at the same time, similar investigations were also carried out by ALEXANDRE-THÉOPHILE VANDERMONDE).

If one considers the three solutions $x_1,\ x_2,\ x_3$ of a cubic equation, then one can number these three solutions in 3! = 6 ways. However, the LAGRANGE resolvent *R* with $R = (1 \cdot x_1 + \alpha \cdot x_2 + \alpha^2 \cdot x_3)^3$ only takes on two different values if we permute the numbering of the solutions. Here, $\alpha$ stands for a non-trivial solution of the equation $x^3 = 1$. These two values then simplify the solution procedure, so that only two auxiliary equations have to be solved to arrive at the solutions of the cubic equation.

For equations of degree 4, the approach with $R = (x_1 + x_2 - x_3 - x_4)^2$ does not lead to 4! = 24, but only to three different values, so that only three auxiliary equations have to be solved for the concrete solution of the equation of degree 4.

For equations of higher degree, however, every resolvent approach proved to be useless. These approaches did not lead to a simplification of the solution procedure. For example, the resolvent for a 5th degree equation takes on six different values – so the situation gets worse.

It seemed that no mathematician before RUFFINI dared to draw the conclusion from this insight:

- *For equations of higher than 4th degree, there is no general solution procedure in radicals.*

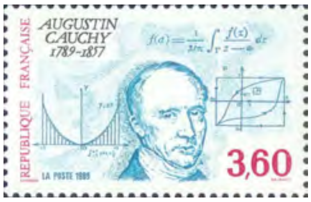
RUFFINI recognised that this was related to the structure of the solution sets. To this end, he was the first to systematically investigate sets of permutations and discovered the first group-theoretical theorems, such as divisibility between the number of elements of groups and their subgroups. In 1799 he published the work *Teoria generale delle equazioni*, in which he set out his insights and gave the first proof of his claim.

He sent a copy of the work to his "compatriot" JOSEPH-LOUIS LAGRANGE (the Turin-born mathematician who originally bore the name GIUSEPPE LODOVICO LAGRANGIA), but LAGRANGE did not respond. RUFFINI suspected that the book had not reached LAGRANGE because of the uncertain political circumstances and sent him a second copy. But again LAGRANGE did not reply.

In 1803 RUFFINI published a second, more rigorous and – as he hoped – easier to understand proof. Initial positive feedback encouraged him to make further improvements regarding the arguments in the proof (he published five versions in total). Others, such as GIANFRANCESCO MALFATTI, who had found a solution method for some special equations of the 5th degree in 1770, did not understand his proof.

RUFFINI at least achieved the result that the *Académie des Sciences* commissioned the board members ADRIEN-MARIE LEGENDRE, SYLVESTRE LACROIX and LAGRANGE to check RUFFINI's proof. Only LAGRANGE gave an opinion: The writing contained too little that was worthy of closer examination.

At least Ruffini received feedback from the *Royal Society* that his proof *probably* proved what he claimed. But although the content of his discovery was actually sensational, this remained without further consequences.




The only person to make a positive statement during Ruffini's lifetime was, of all people, Augustin Cauchy, who usually never has any praise for the achievements of others. He considered the proof to be without any gaps. It was presumably Ruffini's work that inspired Cauchy to his important writings on permutation groups in 1813 and 1815.

After Napoleon's defeat, Ruffini resumed his activities at the university, lectured in mathematics and medicine, and was temporarily rector of the university. When a typhus epidemic swept through the country in 1817, he devoted himself entirely to the care of his patients. In the process, he infected himself, and despite a temporary recovery, he had to gradually give up his work as a university lecturer.

In 1820 he published a treatise on typhoid fever and the effects of the disease as observed in his own body (*Memoria del tifo contagioso*). In 1822 he was infected again – this time with a fatal outcome.

The fact that Ruffini's mathematical work was not adequately appreciated during his lifetime and was even temporarily forgotten after his death was certainly related to the fact that he wrote in Italian. Also, the time was not yet ripe for the revolutionary idea that the search for an algebraic solution method for equations higher than $4^{th}$ degree was futile because such a method could not exist. And the fact that the proof, which was difficult to read and incomplete from today's point of view, was essentially based on the consideration of the structure of permutation groups, which had not been investigated until then, was also a hindrance.

Even Nils Henrik Abel, who did not know Ruffini's writing, had enormous difficulties in 1824 in bringing his proof of the theorem, which today is rightly called the theorem of Abel-Ruffini, to the attention of the mathematicians of his time. Abel's work, too, only gained recognition after his death.



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