Substitution decompositions and pattern classes of permutations

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University of St Andrews

Prologue: From Iceland to England, Mathematics...

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Pastor Jón says that perhaps one gets closest to the creation of the world in mathematical formulae, but adds: Unfortunately I don't know any mathematics.

(*H. Laxness*)

. . . and Permutations

He took the biscuits carefully out of the packet and laid them face upward on the grass, in order as he felt of edibility. They were the same as always, a Ginger, an Osborne, a Digestive, a Petit Beurre and one anonymous. He always ate the first-named last, because he liked it the best, and the anonymous first, because he thought it very likely the least palatable. The order in which he ate the remaining three was indifferent to him and varied irregularly from day to day. On his knees now before the 5 it struck him for the first time that this reduced to a paltry six the number of ways in which he could make his meal. (. . .) Even if he conquered his prejudice against the anonymous, still there would be only twenty-four ways in which the biscuits could be eaten. But were he to take the final step and overcome his infatuation with the ginger, then the assortment would spring to life before him, dancing the radiant measure of its total permutability, edible in a hundred and twenty ways!

(*S. Beckett*)

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Specific results: M.D. Atkinson, NR, R. Smith (Brockport, USA), Substitution-closed pattern classes, submitted for publication.

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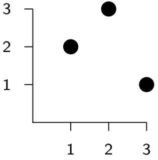

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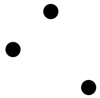

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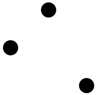

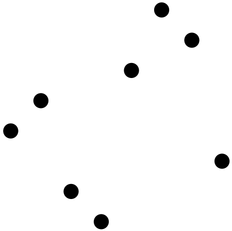

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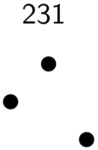

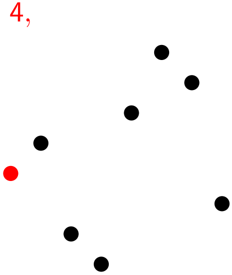

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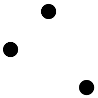

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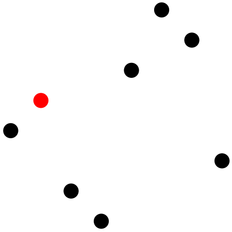

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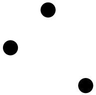

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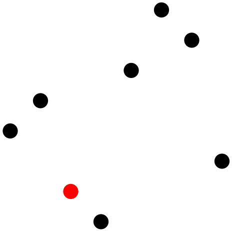

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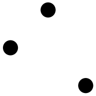

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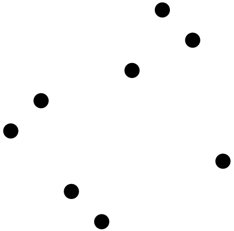

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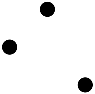

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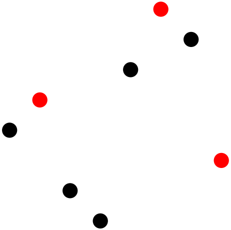

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- $\sigma \preceq \tau$ (σ is involved in τ) if τ contains a subsequence order isomorphic to σ.
- If $\sigma \not\preceq \tau$ say τ avoids σ.

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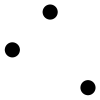

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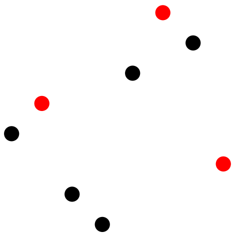

Pattern Classes & Bases

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- Example: $\mathrm{Av}(21) = \{1, 12, 123, 1234, \ldots\}$.

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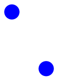

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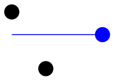

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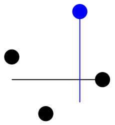

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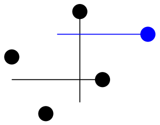

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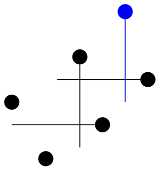

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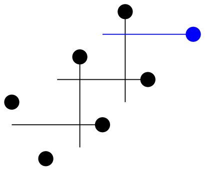

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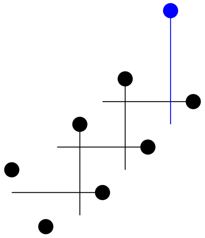

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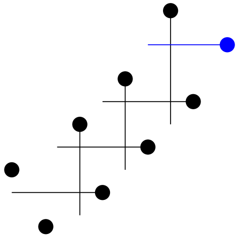

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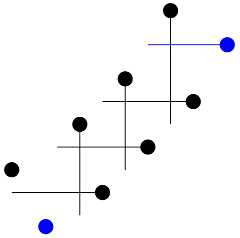

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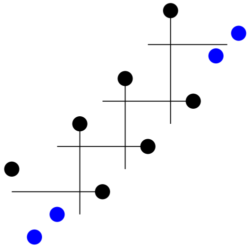

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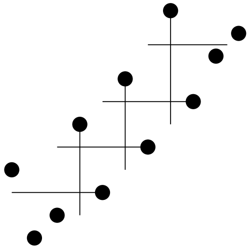

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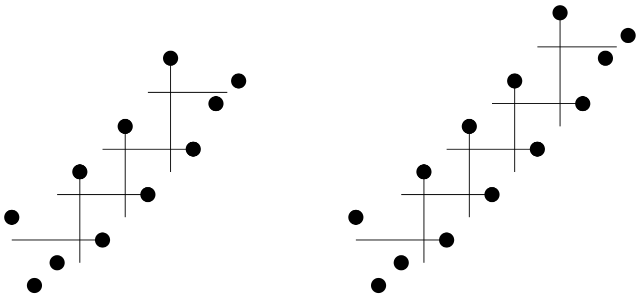

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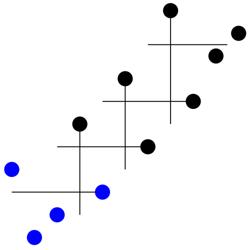

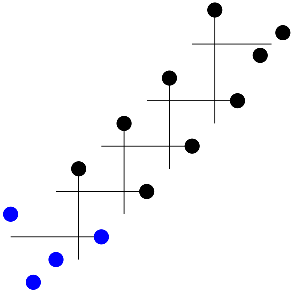

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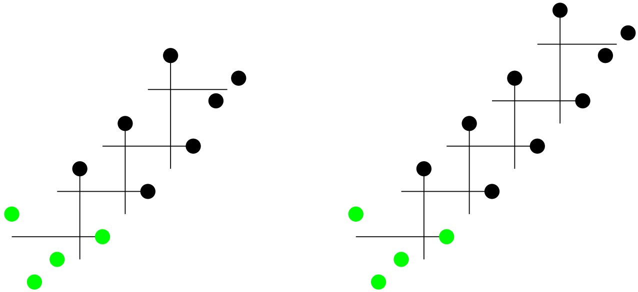

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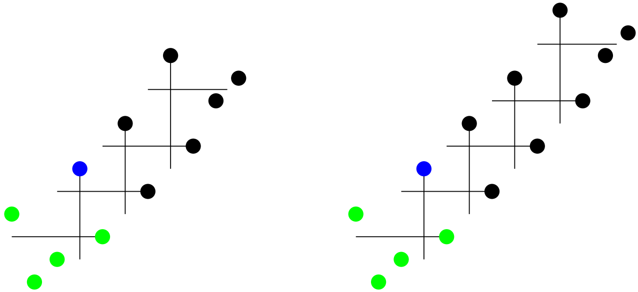

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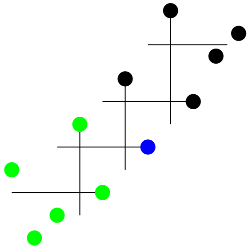

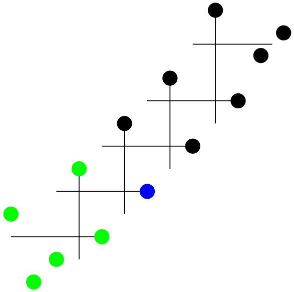

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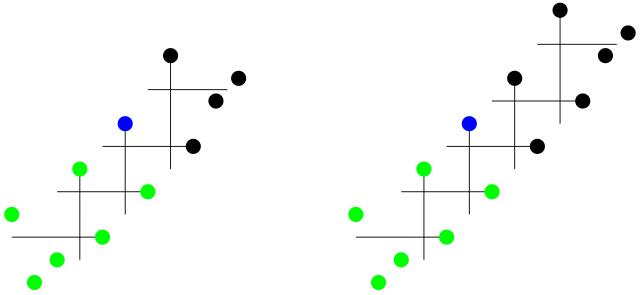

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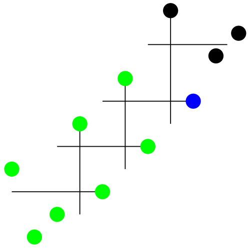

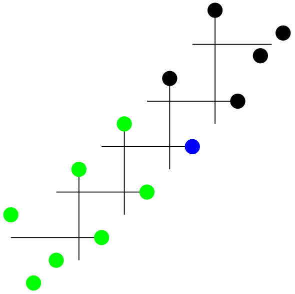

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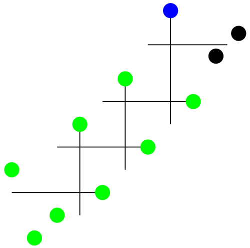

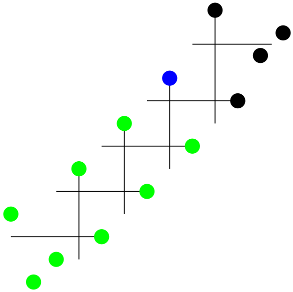

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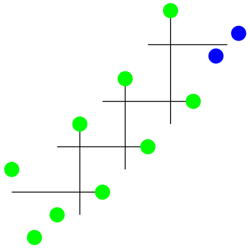

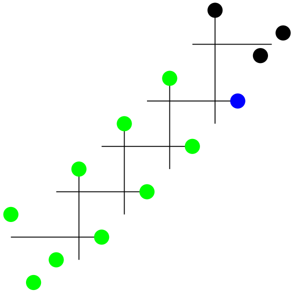

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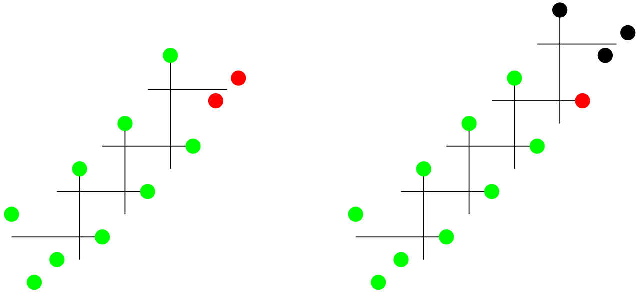

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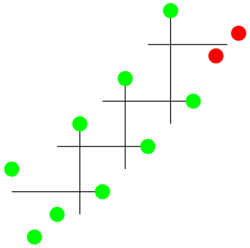
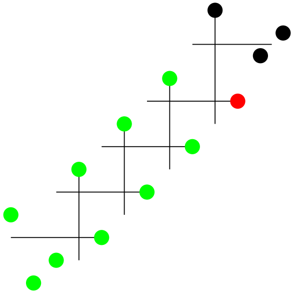
- Hence there are infinitely based pattern classes.





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- Structure: of permutations in $\mathcal{P}$; of $\mathcal{P}$ itself.

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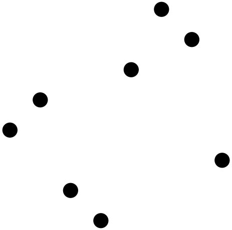

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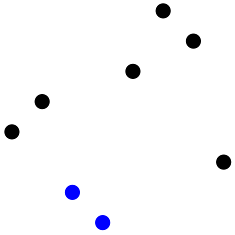

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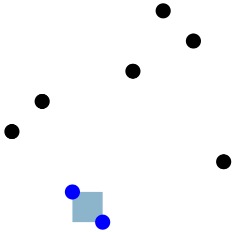

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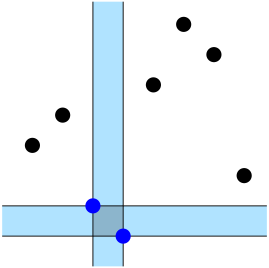

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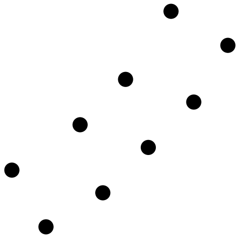

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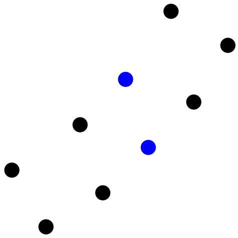

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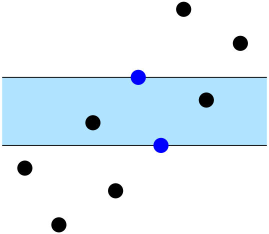

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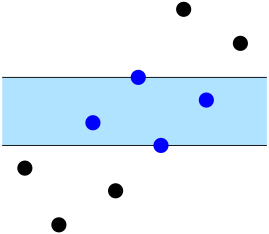

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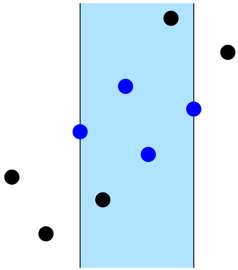

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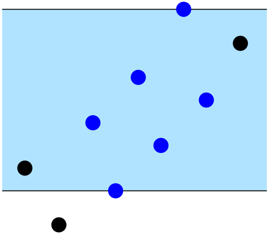

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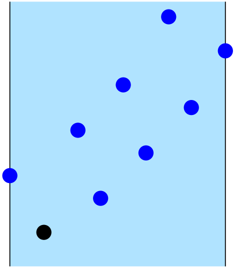

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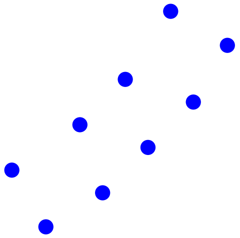
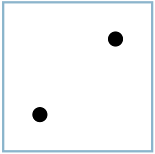
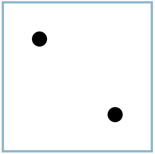


Short Simple Permutations


12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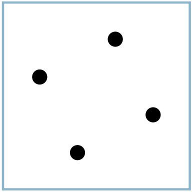
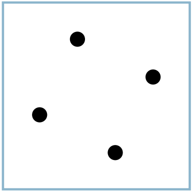
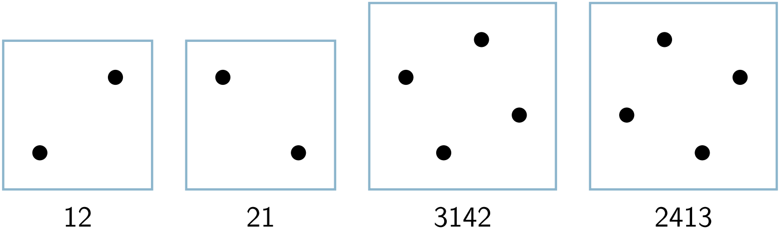

3142


2413

Short Simple Permutations



- Length 5: 6; Length 6: 46; Length 7: 338.

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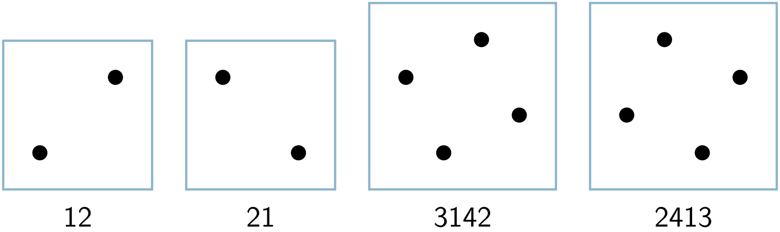

- Length 5: 6; Length 6: 46; Length 7: 338.
- As $n \to \infty$, roughly $1/e^2$ permutations are simple.

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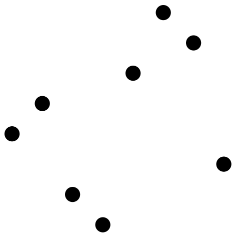

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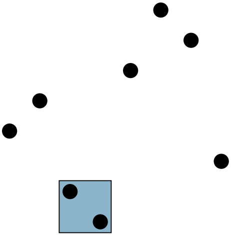

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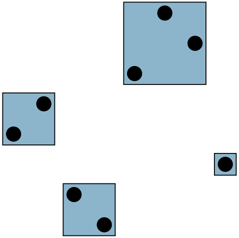

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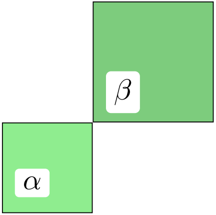

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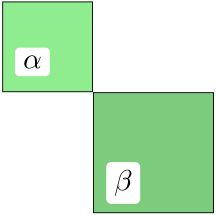

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Modular Decomposition

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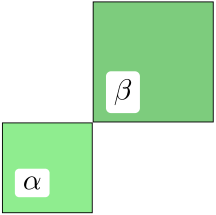
- when σ is $\oplus$-decomposable or $\ominus$-decomposable.
- For a permutation σ define its $\oplus$-length to be the largest k such that $\sigma = \sigma_1 \oplus \ldots \oplus \sigma_k$.



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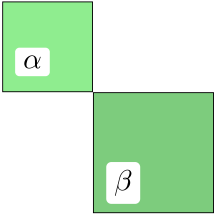

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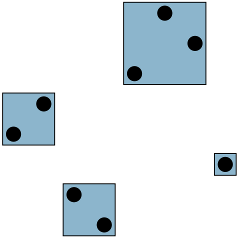

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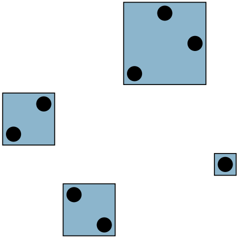

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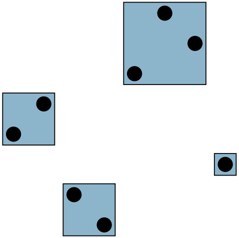

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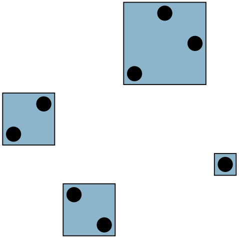

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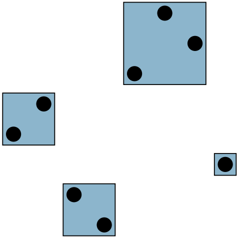

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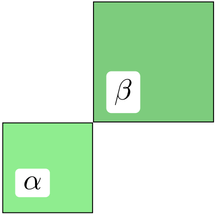

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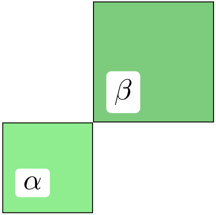

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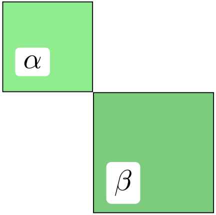

$$= \alpha \ominus \beta = 21[\alpha, \beta]$$

Substitution Closed Classes

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Definition

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Proof

If $\sigma[\tau_1, \dots, \tau_n] \in \mathcal{B}(\mathcal{P})$, then $\sigma, \tau_1, \dots, \tau_n \in \mathcal{P}$ but $\sigma[\tau_1, \dots, \tau_n] \notin \mathcal{P}$.

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Corollary

Suppose $\mathcal{P} = \mathrm{Av}(\sigma)$. *The basis of* $\mathcal{SC}(\mathcal{P})$ *consists of all minimal simple extensions of* σ.

Example

Proposition

$\mathcal{B}(\mathcal{SC}(\mathsf{Av}(231))) = ?$

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$\mathcal{B}(\mathcal{SC}(\mathsf{Av}(231))) = \{2413, 3142\}$.

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$\mathcal{B}(\mathcal{SC}(\mathsf{Av}(231))) = \{2413, 3142\}$.

Proof

- 231 is not simple.
- 2431 and 3142 are, and contain 231.
- Every longer simple permutation containing 231 contains at least one of 2413 or 3142.

The Basis Problem: Formulation

Problem

If a class $\mathcal{P}$ is given by its (finite) basis B, determine whether the basis for its substituion closure is finite or infinite.

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Given a permutation σ determine whether $\mathcal{B}(\mathcal{SC}(\mathsf{Av}(\sigma)))$ is finite or infinite.

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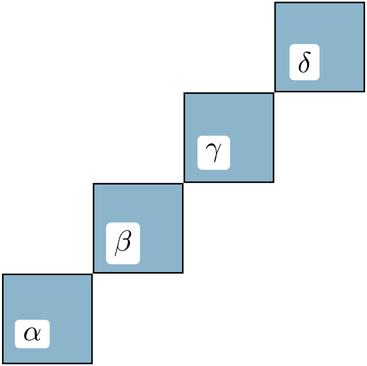
- The basis question for substitution closure in graphs: Giakoumakis (1997); Zverovich (2005).

$\oplus$-decomposable permutations: $\oplus$-length ≥ 4



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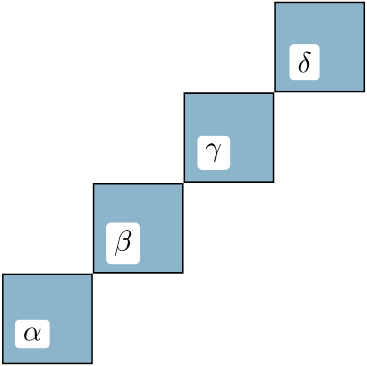

Theorem

Let $\alpha, \beta, \gamma, \delta$ be arbitrary, and $\sigma = \alpha \oplus \beta \oplus \gamma \oplus \delta$. Then the basis of $\mathcal{SC}(\mathsf{Av}(\sigma))$ is infinite.

$\oplus$-decomposable permutations: $\oplus$-length 3

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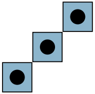

123

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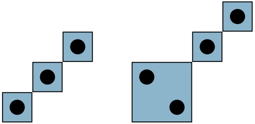

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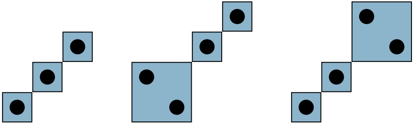

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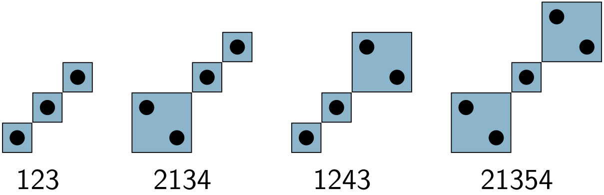

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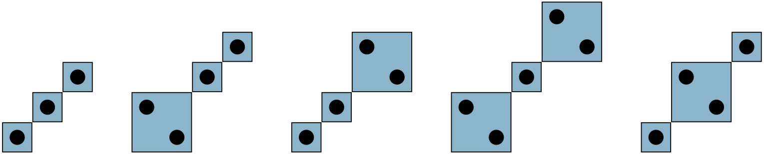

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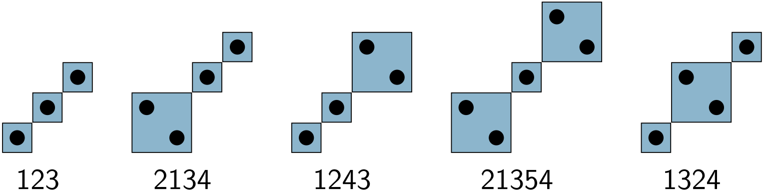

the basis of $\mathcal{SC}(\mathrm{Av}(\sigma))$ is finite. For any other $\sigma = \alpha \oplus \beta \oplus \gamma$ the basis is infinite.

Intermezzo: Proving a Basis Infinite

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Proposition

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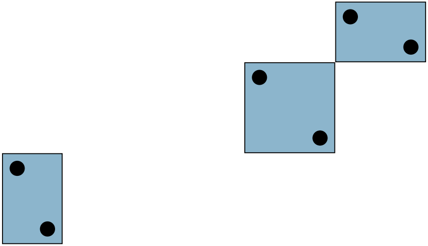

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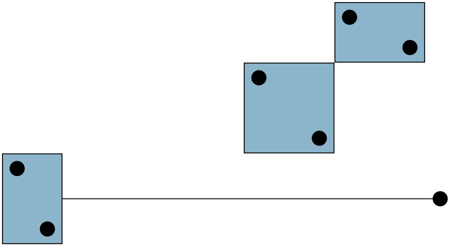

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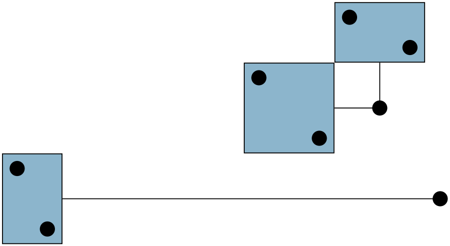

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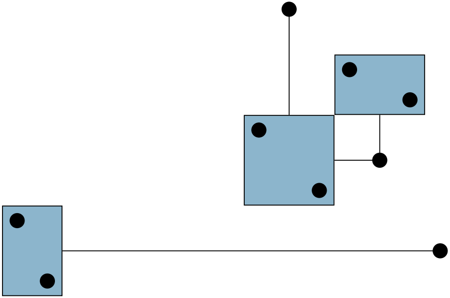

Intermezzo: Proving a Basis Infinite

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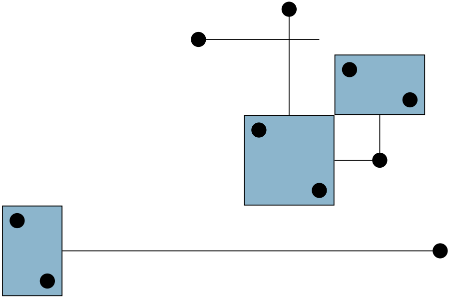

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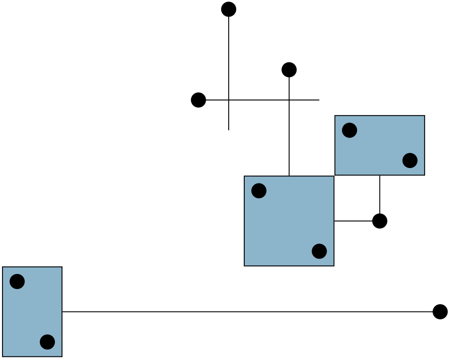

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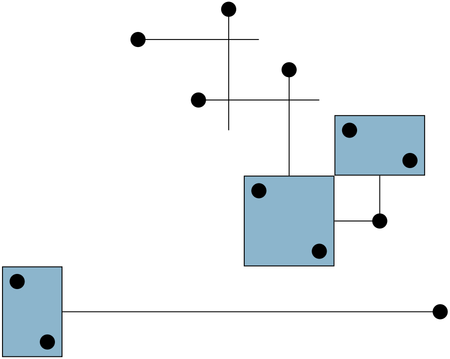

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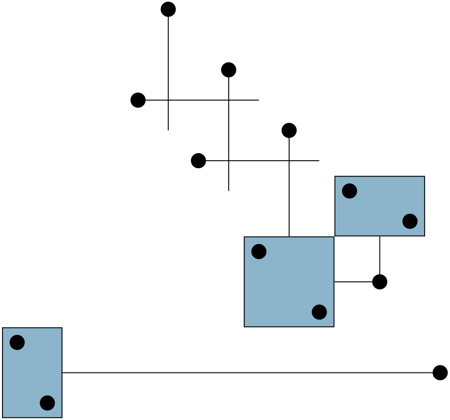

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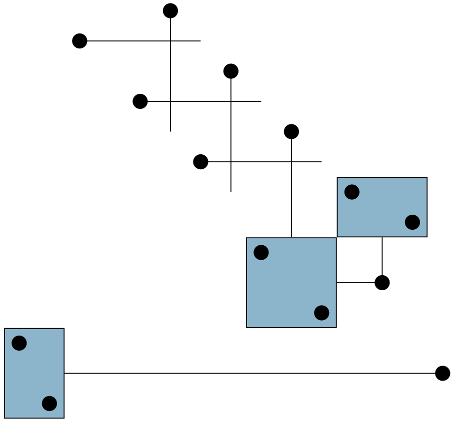

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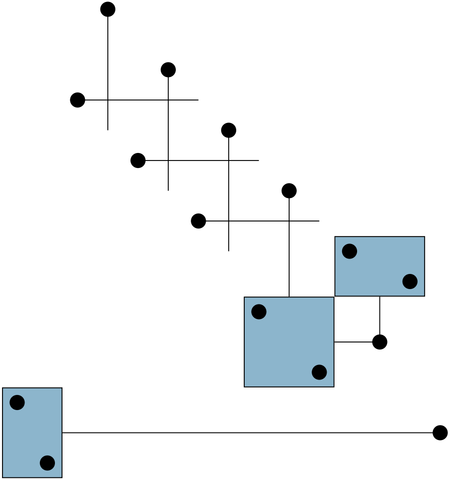

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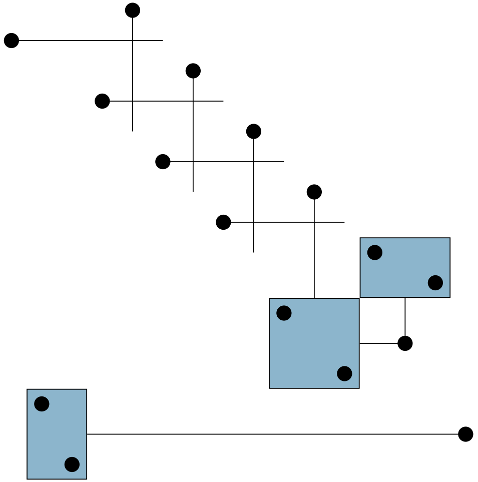

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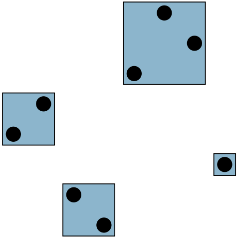
Indecomposable: General Case



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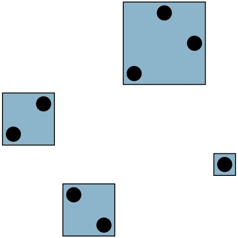

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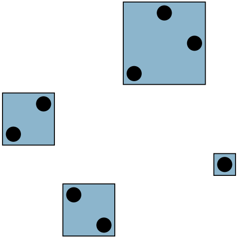

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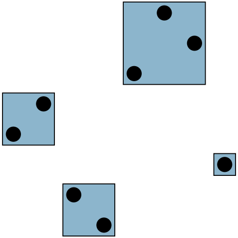

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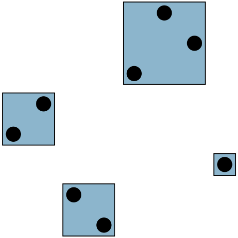

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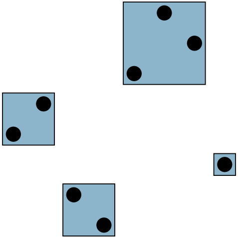

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Except . . .

Spiral Permutations

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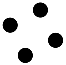

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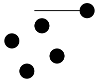

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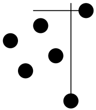

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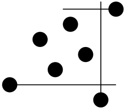

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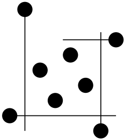

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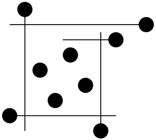

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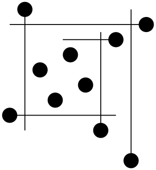

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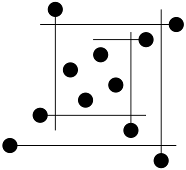

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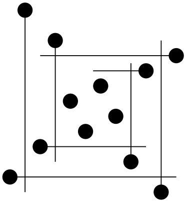

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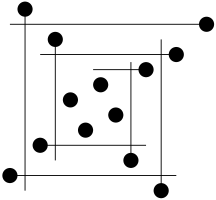

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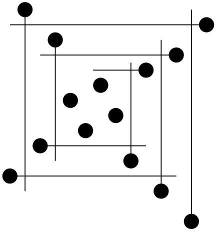

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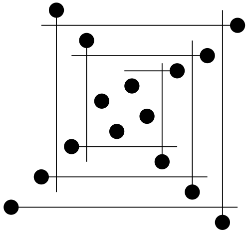

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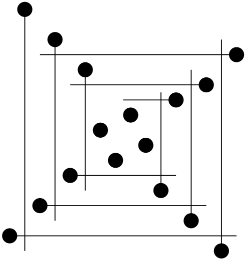

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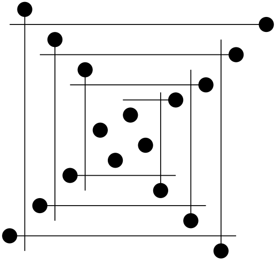

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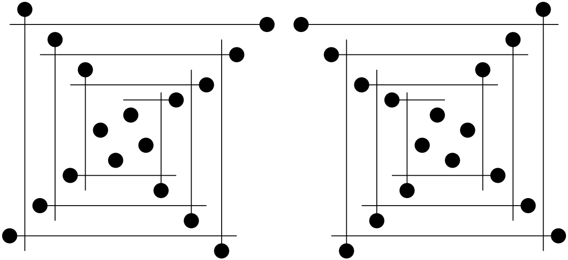

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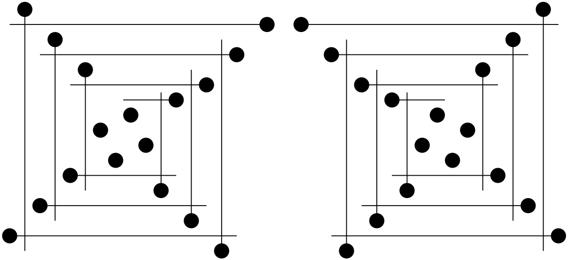

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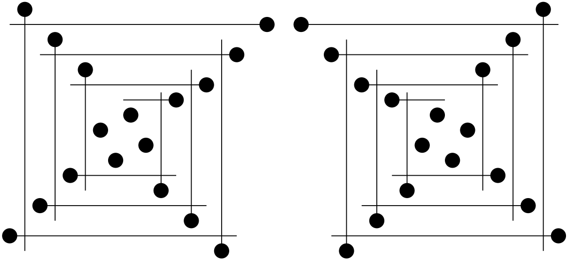

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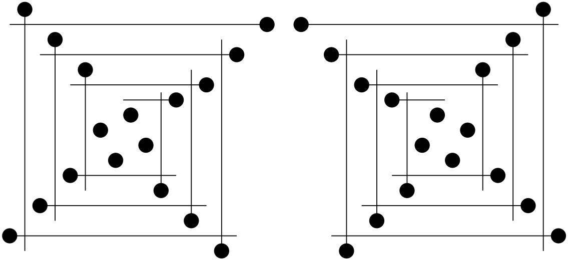

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THIS COMPLETES THE CLASSIFICATION!!!

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4132	41352, 35142, 263514, 531642, 264153, 526413, 362514
4231	463152, 364152, 264153, 536142, 531642, 531462 462513, 362514, 263514, 526413, 524613, 524163 526314, 426315, 513642, 362415, 461352, 416352
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Example

2134 has finite type, but $\mathcal{SC}(\mathsf{Av}(2134, 1234))$ has an infinite basis.

Thank You!

. . . in the certainty that more would not be said however much was added. . .

H. Laxness