

Decidability Questions for Pattern Avoidance Classes of Permutations

Nik Ruškuc, St Andrews

nik@mcs.st-and.ac.uk

<http://www-groups.mcs.st-and.ac.uk/~nik>

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Template Question

Does there exist an algorithm which does the following:

INPUT: A (finite)

(def)

defining a pattern avoidance class X of permutations.

OUTPUT: TRUE if X is

(property)

FALSE otherwise?

If such an algorithm exists we say that the property is decidable; if not, it is undecidable

Algorithms

Algorithm

= Turing machine

= effective computation

= computer program

= unambiguous recipe

- know one when you see it;
- approach borrowed from algebra;
- a fresh look at the area;
- ignore technical issues: complexity, efficiency, etc (for the moment);
- computational tools would be useful nevertheless.

Permutations as Patterns

Sequence = a finite list of distinct numbers.

Permutation = a sequence of numbers $1, \dots, n$ of length n .

Order isomorphism: for sequences $\sigma = s_1 \dots s_m$, $\tau = t_1 \dots t_n$ we say

$$\sigma \cong \tau \Leftrightarrow m = n \ \& \ (\forall i, j : s_i \leq s_j \Leftrightarrow t_i \leq t_j).$$

Example. $8463 \cong 4231$, $8463 \not\cong 4321$.

Fact. For every sequence σ there is a unique permutation $\bar{\sigma}$ such that $\sigma \cong \bar{\sigma}$.

Example. $\overline{8463} = 4231$.

Involvement and Avoidance

Definition. $\sigma = s_1 \dots s_m$ is involved in $\tau = t_1 \dots t_n$ if τ contains a subsequence order isomorphic to σ :

$$\sigma \preceq \tau \Leftrightarrow \exists 1 \leq i_1 < i_2 < \dots < i_m \leq n : t_{i_1} \dots t_{i_m} \cong \sigma.$$

If $\sigma \not\preceq \tau$ we say that τ avoids σ .

Example. $123 \preceq 32415$, $123 \not\preceq 35421$.

Fact. \preceq is a preorder (RT) on the set T of all sequences, and is a (partial) order (RST) on the set S of all permutations.

Fact. $(T, \preceq) / \{(\sigma, \tau) : \sigma \preceq \tau \ \& \ \tau \preceq \sigma\} \cong (S, \preceq)$.

Closed (=Pattern Avoidance)

Classes

Definition. A set $X \subseteq S$ of permutations is said to be closed if

$$\sigma \in X \ \& \ \tau \preceq \sigma \Rightarrow \tau \in X.$$

Another name: pattern avoidance class (why: see later).

Examples.

- S ;
- $I = \{1, 12, 123, \dots\}$;
- $R = \{1, 21, 321, \dots\}$;
- if X and Y are closed then so is $X \cup Y$.

Defining Pattern Classes

- basis;
- token passing networks (TPNs);
- constructions:
 - union;
 - direct sum;
 - juxtaposition;
 - merge;
 - composition;
 - ...
- Sub() operator.

Properties

Definition/representation:

- finitely based;
- generated by a TPN;
- defined by Sub().

Enumeration:

- (a nice) recurrence;
- rational/algebraic/D-finite generating function;
- growth rate.

Structure:

- finite
- sum complete;
- atomic (union indecomposable);
- partially well ordered;
- ...

Basis

Definition. The basis $\mathcal{B}(X)$ of a closed class X is the set of minimal elements not in X :

$$\mathcal{B}(X) = \{\sigma \notin X : \forall \tau : \tau \prec \sigma \Rightarrow \tau \in X\}.$$

Example. $\mathcal{B}(S) = \emptyset$; $\mathcal{B}(I) = \{21\}$; $\mathcal{B}(I \cup R) = \{132, 213, 231, 312\}$.

Definition. For a set $Z \subseteq S$ its avoidance set is

$$\mathcal{A}(Z) = \{\sigma : \forall \tau \in Z : \tau \not\prec \sigma\}.$$

Facts.

- $\mathcal{B}(X)$ is an antichain.
- $\mathcal{A}(Z)$ is closed.
- $\mathcal{A}(\mathcal{B}(X)) = X$.

Basis \rightarrow Finite

Theorem. A closed class X is finite iff its basis contains $12\dots m$ and $n\dots 21$ for some m and n . (And hence finiteness is decidable from the basis.)

Proof. (\Leftarrow) Erdős, Szekeres.

(\Rightarrow) X finite $\Rightarrow \exists m : 12\dots m \notin X$. Choose the smallest such m . Then $12\dots m \in \mathcal{B}(X)$.

Direct Sum

Definition. For $\sigma = s_1 \dots s_m$ and $\tau = t_1 \dots t_n$ define

$$\sigma \oplus \tau = s_1, \dots, s_m, t_1 + m, \dots, t_n + m.$$

Example. $132 \oplus 213 = 132546.$

Definition. $X \oplus Y = \{\sigma \oplus \tau : \sigma \in X, \tau \in Y\}.$

Fact. X, Y closed $\Rightarrow X \oplus Y$ closed.

Examples. $I \oplus I = I; R \oplus R = \mathcal{A}(123, 312, 231).$

Definition. X is sum complete if $\sigma, \tau \in X \Rightarrow \sigma \oplus \tau \in X.$

Examples. I is sum complete, R is not.

Basis \rightarrow Sum Complete

Definition. σ is sum indecomposable if $\sigma \neq \tau \oplus \pi$ for any (non-empty) τ, π .

Theorem. A closed class X is sum complete iff every element of its basis is sum indecomposable.

Proof. (\Rightarrow) Suppose $\sigma = \tau \oplus \pi \in \mathcal{B}(X)$. Then $\tau, \pi \prec \sigma$ implies $\tau, \pi \in X$, but $\tau \oplus \pi \notin X$.

(\Leftarrow)

$$\begin{aligned} & \sigma, \tau \in X \\ \Leftrightarrow & \forall \beta \in \mathcal{B}(X) : \beta \not\leq \sigma \ \& \ \beta \not\leq \tau \\ \Rightarrow & \forall \beta \in \mathcal{B}(X) : \beta \not\leq \sigma \oplus \tau \\ \Leftrightarrow & \sigma \oplus \tau \in X. \end{aligned}$$

Sub() Operator

Definition. Let A, B be (usually infinite) linearly ordered sets, and let $\pi : A \rightarrow B$ be a bijection. For every finite $C \subseteq A$, $\pi|_C$ is order isomorphic to a permutation. The set of all such permutations is denoted by $\text{Sub}(\pi : A \rightarrow B)$ or just $\text{Sub}(\pi)$.

Fact. $\text{Sub}(\pi)$ is closed.

Example. $\pi : \mathbb{N} \rightarrow -\mathbb{N}$, $\pi(x) = -x$. $\text{Sub}(\pi) = R$.

Example. $\pi : \mathbb{N} \rightarrow \mathbb{N}$, $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 3$, $4 \mapsto 4$, ... $\text{Sub}(\pi) = \{12\dots n : n \geq 1\} \cup \{2134\dots n : n \geq 2\}$.

Unions, Sub and Join Property

Theorem. [Atkinson, Murphy, NR] The following conditions are equivalent for a closed class X :

- (i) $X \neq Y \cup Z$ for any proper subclasses Y, Z .
- (ii) $X = \text{Sub}(\pi : A \rightarrow B)$ for some π, A, B .
- (iii) X has the join property:
 $\forall \sigma, \tau \in X : \exists \pi \in X : \sigma \preceq \pi \ \& \ \tau \preceq \pi$.

Definition. A class satisfying one (and hence all) of the above conditions is said to be atomic.

Unions, Sub and Join Property

Example. I , R , $\text{Sub}(2134\dots)$ and $R \oplus R$ are atomic. $I \cup R$ is not atomic.

Fact. Sum complete \Rightarrow atomic.

Open Question. Is atomicity decidable from the basis?

Sub($\pi : \mathbb{N} \rightarrow \mathbb{N}$) (Natural Classes)

Fact. Sum complete \Rightarrow natural.

Fact. $\gamma \in S$, X sum complete \Rightarrow $\text{Sub}(\gamma) \oplus X$ natural.

Theorem. [Atkinson, Murphy, NR] If X is a finitely based natural class then one of the following holds:

- (i) $X = \text{Sub}(\gamma) \oplus Y$, where $\gamma \in S$, Y is sum complete and uniquely determined by X ; or
- (ii) $X = \text{Sub}(\pi : \mathbb{N} \rightarrow \mathbb{N})$ where π is uniquely determined by X and is ultimately periodic:

$$\exists N, \omega : \forall n \geq N : \pi(n + \omega) = \pi(n) + \omega.$$

Basis \rightarrow Natural

Corollary. [Murphy] It is decidable whether a finitely based closed class is natural.

Open Questions. Is it decidable whether a finitely based class can be written as any of the following:

- $\text{Sub}(\pi : \mathbb{Z} \rightarrow \mathbb{Z})?$
- $\text{Sub}(\pi : 2\mathbb{N} \rightarrow \mathbb{N})?$
- $\text{Sub}(\pi : \mathbb{Q} \rightarrow \mathbb{Q})?$

Enumeration

$s_n(X)$ = the number of permutations of length n in X .

Example. $s_n(I) = s_n(R) = 1$; $s_n(R \oplus R) = n$.

Theorem. [Knuth; Simion, Schmidt] If π is a permutation of length 3 then $s_n(\mathcal{A}(\pi)) = C_n$, the n th Catalan number.

Remarks. Exact formulae are known for $s_n(\mathcal{A}(\pi))$ where π is any permutation of length 4, except when π is (equivalent to) 1324. Very little is known about $s_n(\mathcal{A}(\pi))$ when $|\pi| \geq 5$.

Enumeration: Generating Functions

Open Questions. Is it decidable whether the ordinary generating function of $s_n(X)$ for a finitely based closed class X is (a) rational? (b) algebraic? (c) D -finite?

Theorem. [Bousquet-Melou] The generating function for $\mathcal{A}(1234)$ is D -finite but not algebraic.

Theorem. [Gessel] Is the generating function for a finitely based closed class always D -finite?

Theorem. [Murphy] There exists an infinitely based closed class the generating function of which is not D -finite.

Enumeration: Growth

Theorem . If $X \neq S$ is a closed class then there exists q such that $s_n(X) \leq q^n$.

Remarks. Proved very recently by Marcus and Tardos. Before that it was known as the Wilf–Stanley Conjecture.

Corollary. The limit $q = \lim_{n \rightarrow \infty} \sqrt[n]{s_n(X)}$ exists. (The growth of X .)

Examples. The growth of $\mathcal{A}(\pi)$ where $|\pi| = 3$ is 4. [Regev] The growth of $\mathcal{A}(12 \dots k)$ is $(k - 1)^2$.

Questions. For a fixed q , is it decidable whether q is the growth of a finitely based class X ? Is the growth of a finitely based class effectively computable?

Remark. [Bona] The growth of $\mathcal{A}(12453)$ is $9 + 4\sqrt{2}$.

Token Passing Networks (TPNs)

A TPN is a finite directed graph with a distinguished input vertex I and a distinguished output vertex O . Each vertex is one of the following:

- simple node, capable of holding one item of data;
- stack, capable of holding any number of items, and treating them in the FILO discipline;
- queue (FIFO);
- ...

If all vertices are of type (i) we have a finite capacity (FC) TPN.

TPNs

Let N be a TPN. N can generate permutations: feed $12\dots n$ into N , item by item, through I , move items along edges respecting the orientation, store them in vertices respecting the type, output them via O .

$\mathcal{P}(N) =$ the set of all output permutations.

Facts. $\mathcal{P}(N)$ is closed, sum complete and atomic.

Remark. Now atomicity is decidable!

TPN: Example

$$X = \mathcal{P}(N)$$

- X is not finitely based.
- Unknown: $\mathcal{B}(X)$; $s_n(X)$; growth; enumeration for the basis.

FCTPNs

Let N be a FCTPN, $|N| = m$, $X = \mathcal{P}(N)$.

Theorems. (i) [Atkinson, Livsey, Tulley] Elements of N can be encoded, symbol by symbol, by words over an alphabet of size at most m . [The actual size of the alphabet will be called the boundedness of X .] The resulting set of words is a regular language.

(ii) The generating function for X is rational and can be effectively computed.

(iii) [Albert, Atkinson, NR] $\mathcal{B}(X)$ can be encoded in the same fashion, over the same alphabet, again yielding a regular language.

(iv) It is decidable whether X is finitely based.

Remark. The above algorithms are practical and have been implemented in GAP.

FCTPNs

Theorem. [Albert, Linton, NR] For any m there are only finitely many pattern classes of the form $\mathcal{P}(N)$, where N is a FCTPN of bound-
edness m .

Fact. If a class X is given by its basis, it is decidable whether X is bounded.

Question. Is it decidable whether a given finitely based class is generated by a FCTPN?

TPNs

Question. Is it decidable whether a given TPN generates a finitely based class?

Theorem. [Waton] It is decidable whether a given TPN generates S , the class of all permutations.

Idea. Avoid

Partial Well Order: Towards Undecidability?

Definition. [Following G. Higman] A pattern class X is said to be partially well ordered (PWO) if it contains no infinite antichain.

Theorem. [Atkinson, Murphy, NR] A finitely based closed class is PWO iff it has only countably many subclasses.

Examples.

- $\mathcal{A}(12)$ is PWO.
- [Atkinson, Murphy, NR] $\mathcal{A}(231)$ is PWO.
- [Spielman, Bona] $\mathcal{A}(123)$ is not PWO.

Questions. Is partial well orderedness decidable for closed classes given by (a) finite bases? (b) TPNs? (c) FCTPNs?

```
> f:=proc(list)
> RETURN(true);
> end;
      f := proc(list) RETURN(true) end proc

> f([2,3,1],[3,5,2,1,4]);
      true
```