On the Berman Conjecture on p_n Sequences of Finite Semigroups

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Term Operations

Let $\mathbf{A} = (A, F)$ be an algebra.

Every term $t(x_1, \ldots, x_n)$ induces a function

$$\underbrace{A \times \ldots \times A}_{n} \longrightarrow A,$$

by substitution of elements of A for variables.

We denote this function also by $t(x_1, \ldots, x_n)$ and call it an (n-ary) term operation (or polynomial).

The set of all n-ary term operations of $\mathbf{A}$ is denoted by $\mathcal{T}_n = \mathcal{T}_n(\mathbf{A})$.

For a semigroup: term = word.

Essentiality

A function

$$t = t(x_1, \ldots, x_n) \;:\; \underbrace{A \times \ldots \times A}_{n} \longrightarrow A$$

is said to <u>depend on x_i</u> if there exist

$$a_1, \ldots, a_{i-1}, a_i', a_i'', a_{i+1}, \ldots, a_n \in A$$

such that

$$\begin{aligned} & t(a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_n) \\ \neq \; & t(a_1, \ldots, a_{i-1}, a_i'', a_{i+1}, \ldots, a_n). \end{aligned}$$

t is said to be <u>essentially n-ary</u> if it depends on all $x_1, \ldots, x_n$.

The set of all essentially n-ary term operations of an algebra $\mathbf{A}$ is denoted by $\mathcal{E}_n = \mathcal{E}_n(\mathbf{A})$.

Let $\mathcal{E}_0(\mathbf{A})$ be the set of all unary term operations which do not depend on their only variable.

The p_n-sequence

Definition. The p_n-sequence of $\mathbf{A}$ is

$$\mathbf{p}(\mathbf{A}) = (p_0(\mathbf{A}), p_1(\mathbf{A}), p_2(\mathbf{A}), \ldots)$$

where

$$p_n(\mathbf{A}) = |\mathcal{E}_n(\mathbf{A})|.$$

Example: Semilattices

Let S be the two element semilattice $\{0, 1\}$.

Every term with n variables is equal (in S) to $x_1 x_2 \dots x_n$.

$x_1 x_2 \dots x_n \in \mathcal{E}_n$ (put $a_i' = 0$, $a_i'' = 1$ and $a_j = 1$ for $j \neq i$).

Thus

$$\mathbf{p}(S) = (0, 1, 1, 1, 1, 1, \dots).$$

In fact the above holds for every non-trivial semilattice.

Also, the converse holds: if S is a semigroup with p_n-sequence $(0, 1, 1, 1, \dots)$ then S is a semilattice.

More Examples

Example. If S is a rectangular band then

$$\mathbf{p}(S) = (0, 1, 2, 0, 0, 0, \ldots).$$

(The only essential term operations are x, xy and yx.)

Example. If S is a nilpotent semigroup of class m (i.e. $S^m = \{0\}$) then

$$\mathbf{p}(S) = (p_0, p_1, \ldots, p_{m-1}, 0, 0, 0, \ldots).$$

More Examples

Example. If B is a boolean group (group of exponent 2, written additively) then

$$\mathbf{p}(B) = (1, 1, 1, 1, 1, 1, \ldots).$$

(The only essentially n-ary term operation is $x_1 + x_2 + \ldots + x_n$.)

Example. Let B be a boolean group. Consider the ternary operation $d : B \times B \times B \longrightarrow B$ defined by

$$d(x, y, z) = x + y + z,$$

and let $\mathbf{D} = (B, d)$. Then

$$\mathbf{p}(\mathbf{D}) = (0, 1, 0, 1, 0, 1, \ldots).$$

(The only essentially n-ary term operations are $x_1 + x_2 + \ldots + x_n$ for odd n.)

More Examples

Example. For a semigroup S,

$$\mathbf{p}(S) = (0, 1, 2, 3, 4, 5, \ldots).$$

if and only if S is a left or right normal band.

Example. (Crvenkovic, Ruškuc [8]) For a semigroup S,

$$\mathbf{p}(S) = (0, 1, 4, 9, 16, 25, \ldots).$$

if and only if S is a normal band.

p_n-sequences and Free Spectra

For an algebra $\mathbf{A}$, its free spectrum is

$$\mathbf{f}(\mathbf{A}) = (f_0(\mathbf{A}), f_1(\mathbf{A}), f_2(\mathbf{A}), \ldots)$$

where $f_n(\mathbf{A})$ is the order of the free algebra of rank n in the variety generated by $\mathbf{A}$, i.e. $f_n(\mathbf{A}) = |\mathcal{T}_n(\mathbf{A})|$.

Proposition. For an algebra $\mathbf{A}$ with

$$\mathbf{p}(\mathbf{A}) = (p_0, p_1, p_2, p_3, \ldots),$$
$$\mathbf{f}(\mathbf{A}) = (f_0, f_1, f_2, f_3, \ldots)$$

then

$$f_n = \sum_{i=0}^{n} \binom{n}{i} p_i,$$
$$p_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f_i.$$

Corollary. $p_n(\mathbf{A})$ is finite for all n if and only if $\mathbf{A}$ generates a locally finite variety.

The Berman Conjecture

Conjecture. (J. Berman [1]) The p_n-sequence of a finite algebra is either bounded or eventually strictly increasing.

Example: Monoids

Proposition. The Berman Conjecture holds for monoids.

Proof. Let M be a (non-trivial, finite) monoid.

Claim: The mapping $\phi : \mathcal{E}_n \longrightarrow \mathcal{E}_{n+1}$ defined by $t \mapsto tx_{n+1}$, where $t = t(x_1, \ldots, x_n) \in \mathcal{E}_n$, is a well-defined injection. (First take $x_1 = \ldots = x_n = 1$, and then take $x_{n+1} = 1$.)

Claim. $x_1 \ldots x_n x_{n+1}^2 \in \mathcal{E}_{n+1}$ unless M is a boolean group. (Take $x_1 = \ldots = x_n = 1$.)

Claim. $x_1 \ldots x_n x_{n+1}^2 \neq \phi(t)$ for every $t \in \mathcal{E}_n$, unless M is a band. (Take $x_1 = \ldots = x_n = 1$.)

Claim. If M is a band, but not a semilattice, then $x_{n+1}x_1x_2 \ldots x_n \neq \phi(t)$ for every $t \in \mathcal{E}_n$. (M contains a two element left or right zero subsemigroup $\{e, f\}$. Now take $x_1 = \ldots = x_n = e$, $x_{n+1} = f$.)

A General Positive Result

Definition. A binary operation $\cdot$ is strongly onto if

(1) $(\forall a)(\exists b_1, b_2)(a = b_1 b_2 = b_2 b_1)$.

(2) $(\forall a, b)[(\exists c, d_1, d_2)(a = cd_1 \ \& \ b = cd_2)$
$\iff (\exists c_1, c_2, d)(a = c_1 d \ \& \ b = c_2 d)]$.

Facts.
(1) commutative + onto $\Rightarrow$ strongly onto.
(2) neutral element (identity) $\Rightarrow$ strongly onto.

Theorem. (J. Berman, A. Kisielewicz, [2]) The Berman Conjecture holds for every finite algebra with a strongly onto binary operation.

Remark. Mapping:
$t(x_1, \ldots, x_n) \mapsto t(x_1, \ldots, x_{n-1}, x_n x_{n+1})$.

Corollary. The Berman Conjecture holds for groups, rings, lattices, monoids, commutative surjective ($S^2 = S$) semigroups, etc.

Extending to Infinite?

Theorem. (G. Grätzer, A. Kisielewicz [9]) For every sequence $(p_0, p_1, p_2, p_3, \ldots)$ with $p_0, p_1 > 0$ there exists an algebra $\mathbf{A}$ such that

$$\mathbf{p}(\mathbf{A}) = (p_0, p_1, p_2, p_3, \ldots).$$

Corollary. The Berman Conjecture does not hold if one removes the requirement for the algebra to be finite.

Willard's Counterexample

Theorem. (R. Willard, [10]) There exist finite algebras violating the Berman Conjecture.

Sketch of Proof. Let B be a boolean group, and let $d(x, y, z) = x + y + z$. Also, define $xy = x$.

Let $I = \{0, 1\}$ be the two element semilattice, and define $d(x, y, z) = 0$.

Willard's Counterexample

Let $\mathbf{A} = (B, d, \cdot) \times (I, d, \cdot)$.

Essentially n-ary term operations on $\mathbf{A}$:

$$(f, 0),\ (x_i, g),\ f \in \mathcal{E}_n(B, d),\ g \in \mathcal{E}_n(I).$$

Thus

$$p_n(\mathbf{A}) = p_n(B, d) + np_n(I) = \begin{cases} 1 + n & n \text{ odd} \\ n & n \text{ even.} \end{cases}$$

$\mathbf{p}(\mathbf{A}) = (0, 2, 2, 4, 4, 6, 6, \ldots)$.

A minor modification gives algebras with p_n-sequence $(0, N+1, 2, N+3, 4, N+5, \ldots)$ for arbitrarily large N.

Bounded p_n-sequences

Let S be a finite semigroup.

$\omega = \min\{m \,:\, S^m = S^{m+1}\}$.

Facts. S^ω is surjective ($(S^\omega)^2 = S^\omega$), and S is a nilpotent extension on S^ω.

Theorem. (Crvenkovic, Dolinka, Ruškuc [5]) S has a bounded p_n-sequence iff S^ω is a semilattice, or a rectangular band, or a boolean group.

Bounded p_n-sequences

Sketch of Proof. S has a bounded p_n-sequence iff it satisfies identities of one of the following types:

(1) $x_1x_2\ldots x_m = x_2\ldots x_mx_1$,
$x_1^2x_2x_3\ldots x_n = x_1x_2^2x_3\ldots x_n$.

(2) $x_1x_2\ldots x_m = x_2\ldots x_mx_1$,
$x_1^2x_2x_3\ldots x_n = x_2^3x_3\ldots x_n$.

(3) $x_1\ldots x_kyx_{k+1}\ldots x_n = x_1\ldots x_kx_{k+1}\ldots x_n$.

These correspond to nilpotent extensions of (1) semilattices, (2) boolean groups, (3) rectangular bands.

Surjective semigroups

Theorem. (Crvenkovic, Dolinka, Ruškuc [6]) The Berman Conjecture holds for all finite surjective semigroups.

Sketch of Proof. Define $\phi : \mathcal{E}_n \longrightarrow \mathcal{E}_{n+1}$ by $t(x_1, \ldots, x_n) \mapsto t(x_1, \ldots, x_{n-1}, x_n x_{n+1})$.

Fact. (Chrislock [3]) If T satisfies a heterotypical identity, then T^ω completely simple. Hence ϕ is well-defined.

Surjectivity of S implies that ϕ is 1-1.

Separate treatment for completely simple.

S^ω Completely Regular

Theorem. (Crvenkovic, Dolinka, Ruškuc [4]) If S^ω is completely regular (union of groups) then S satisfies the Berman Conjecture.

Sketch of Proof. Let r be such that $x^{r+1} = x$ for all $x \in S^\omega$.

Define $\phi : \mathcal{E}_n \longrightarrow \mathcal{E}_{n+1}$ by $t \mapsto t^r x_{n+1} t$.

S^ω Reductive

T is (right) reductive if

$$(\forall x \in T)(ax = bx) \Rightarrow a = b$$

for all $a, b \in T$ ($\Longleftrightarrow$ no two rows of the Cayley table are identical).

Theorem. (Crvenkovic, Dolinka, Ruškuc [4]) If S^ω is reductive then S satisfies the Berman Conjecture.

Sketch of Proof. $t \mapsto tx_{n+1}$.

Corollary. The Berman Conjecture holds for inverse semigroups and for commutative semigroups.

S^ω Regular

Theorem. (Crvenkovic, Dolinka, Ruškuc [7]) If S^ω is regular then S satisfies the Berman Conjecture.

Sketch of Proof. Wlog assume that S^ω is not reductive.

Find a 'minimal' pair $a, b \in S^\omega$ witnessing that S^ω is not reductive; it has the following properties:

(1) $a\mathcal{L}b$;

(2) For every $s \in S$ we have

$$(as\mathcal{R}a \ \&\ bs\mathcal{R}b) \text{ or } as = bs.$$

Form the principal congruence

$$\theta = \theta(a, b) = \{(as, bs) \ :\ s \in S^1\} \cup \Delta_S.$$

Basically it identifies R_a and R_b.

S^ω Regular

Case 1. If S^ω/θ is not left reductive then $p_n(S) = p_n(S/\theta)$. Now use an inductive argument.

Case 2. S^ω/θ is left reductive.

Here one proves directly that S satisfies the Berman Conjecture.

To define an injection $\mathcal{E}_n \longrightarrow \mathcal{E}_{n+1}$ proceed as follows.

First define

$T_{n,i} = \{t \in \mathcal{E}_n \ : \ t = x_i w' x_i w''\}$.

$Q_{n,i} = \{t \in \mathcal{E}_n \ : \ t = x_i w'\} \setminus T_{n,i}$.

If $t = x_i w' x_i w'' \in T_{n,i}$ then $t \mapsto x_i w' x_{n+1} w''$.

If $t \in Q_{n,i}$ then $t \mapsto x_{n+1} t$.

What Next?

Question. Can one prove the Berman Conjecture in certain cases by constructing a surjection $\mathcal{E}_{n+1} \longrightarrow \mathcal{E}_n$, instead of an injection $\mathcal{E}_n \longrightarrow \mathcal{E}_{n+1}$?

Problem. Investigate the eventual behaviour of p_n-sequences of infinite (locally finite) semigroups.

References

1. J. Berman, *Free Spectra of Finite Algebras*, Lecture Notes, University of Illinois at Chicago, 1986.

2. J. Berman, A. Kisielewicz, On the number of operations in a clone, *Proc. Amer. Math. Soc.* **122** (1994), 359–369.

3. J.L. Chrislock, A certain class of identities of semigroups, *Proc. Amer. Math. Soc.* **21** (1969), 189–190.

4. S. Crvenkovic, I. Dolinka, N. Ruškuc, Notes on the number of operations of finite semigroups, *Acta Sci. Math. (Szeged)* **66** (2000), 23–31.

5. S. Crvenkovic, I. Dolinka, N. Ruškuc, Finite semigroups with few term operations, *J. Pure Appl. Algebra* **157** (2001).

6. S. Crvenkovic, I. Dolinka, N. Ruškuc, The Berman Conjecture is true for finite surjective semigroups and their inflations, *Semigroup Forum* **62** (2001), 103–114.

7. S. Crvenkovic, I. Dolinka, N. Ruškuc, The Berman Conjecture for nilpotent extensions of regular semigroups, in preparation.

References

8. S. Crvenkovic, N. Ruškuc, On semigroups having n^2 essentially n-ary polynomials, *Algebra Universalis* **30** (1993), 269–271.

9. G. Grätzer, A. Kisielewicz, A survey of some open problems on p_n-sequences and free spectra of algebras and varieties, in *Universal Algebra and Quasigroup Theory*, A. Romanowska, J.D.H. Smith (eds.), Heldermann, Berlin, 1992, 57–88.

10. R. Willard, Essential arities of term operations in finite algebras, *Discrete Math.* **149** (1996), 239–259.