Endomorphism Monoids of Combinatorial Structures

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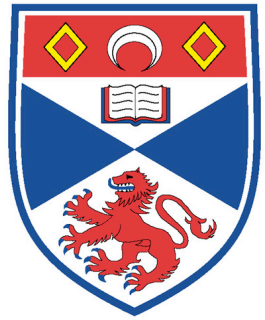

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- Each relation R_i has an arity r_i, meaning $R_i \subseteq X^{r_i}$.
- The sequence $(r_i\ (i \in I))$ is called the signature of $\mathcal{X}$.

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- Posets: signature (2); properties R, AS, T.
- Permutations: signature (2,2); properties: two linear orders.

Morphisms

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Let $\mathcal{X} = (X;\ R_i\ (i \in I))$ be a relational structure. An endomorphism is a mapping $\theta : X \to X$ which respects all the relations R_i, i.e.

$$(x_1, \ldots, x_k) \in R_i \Rightarrow (x_1\theta, \ldots, x_k\theta) \in R_i.$$

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General Problem

For a given $\mathcal{X}$, how are $\mathcal{X}$, $\mathrm{End}(\mathcal{X})$ and $\mathrm{Aut}(\mathcal{X})$, and their properties, related?

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Problem

Describe the monoids $\mathrm{End}(P)$ *where* P *is a permutation.*

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Question

What is $\mathrm{rank}(B_n)$, where B_n is the monoid of all binary relations on $\{1, \ldots, n\}$?

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Example

$\text{rank}(\text{Trans}(n) : \text{Sym}(n)) = 1$.

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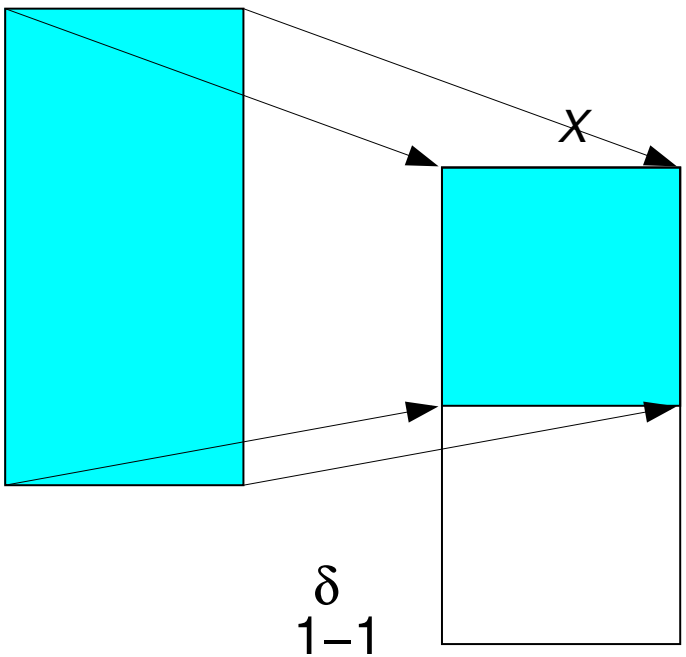

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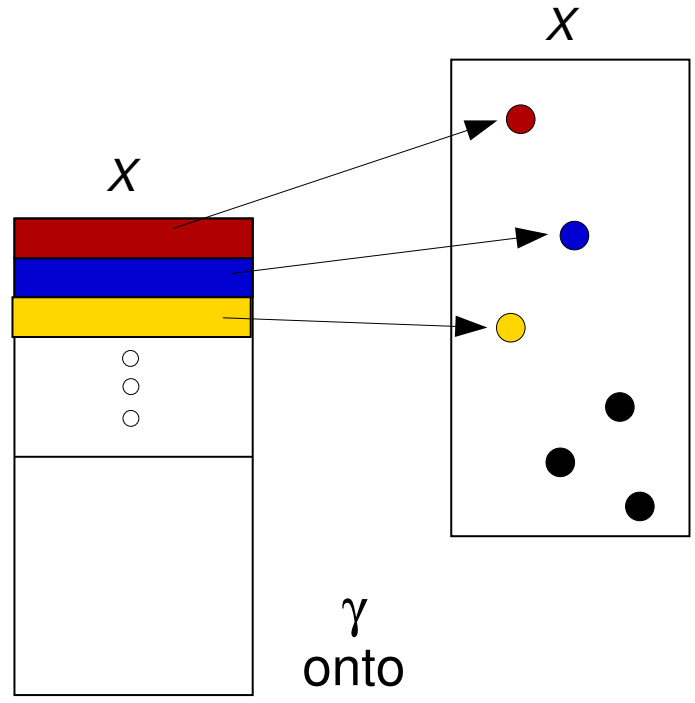

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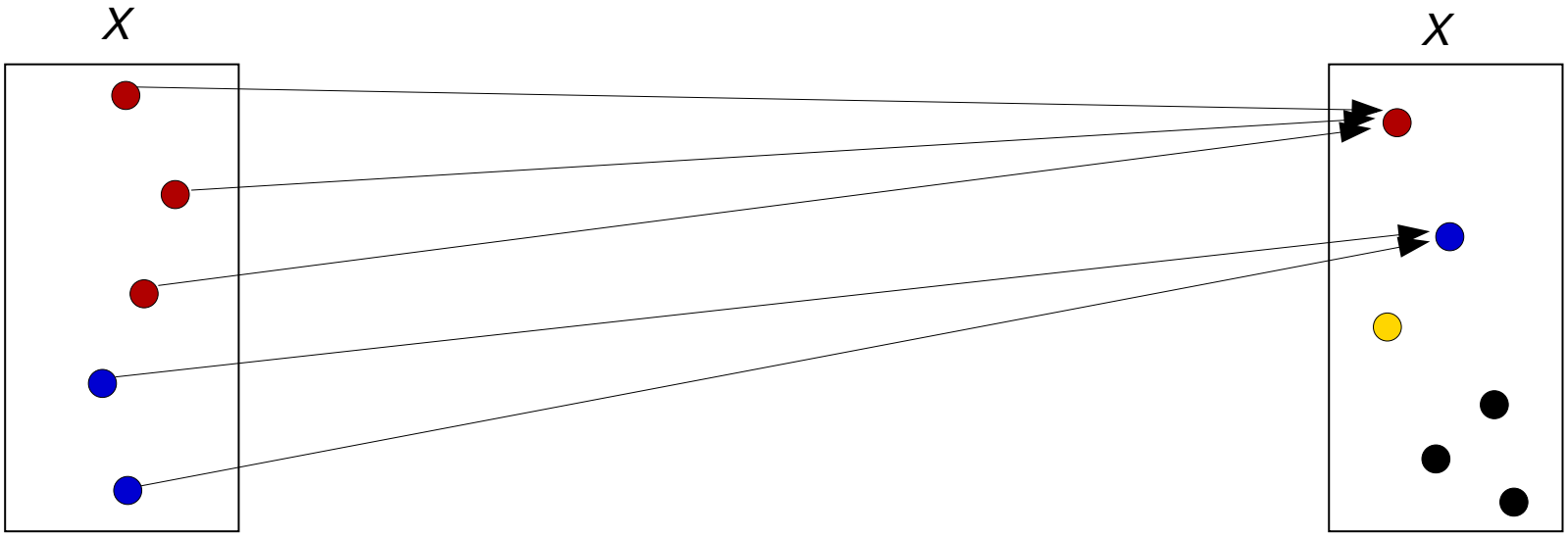

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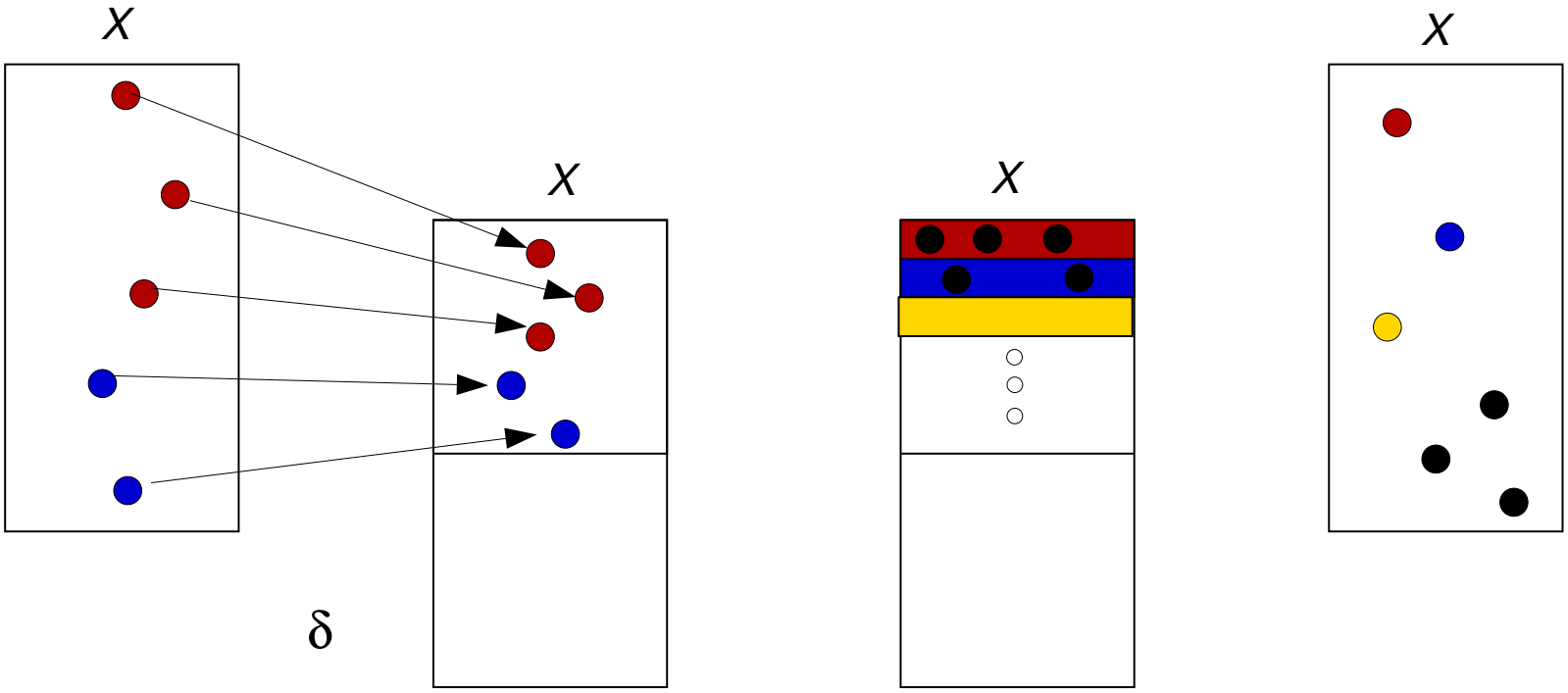

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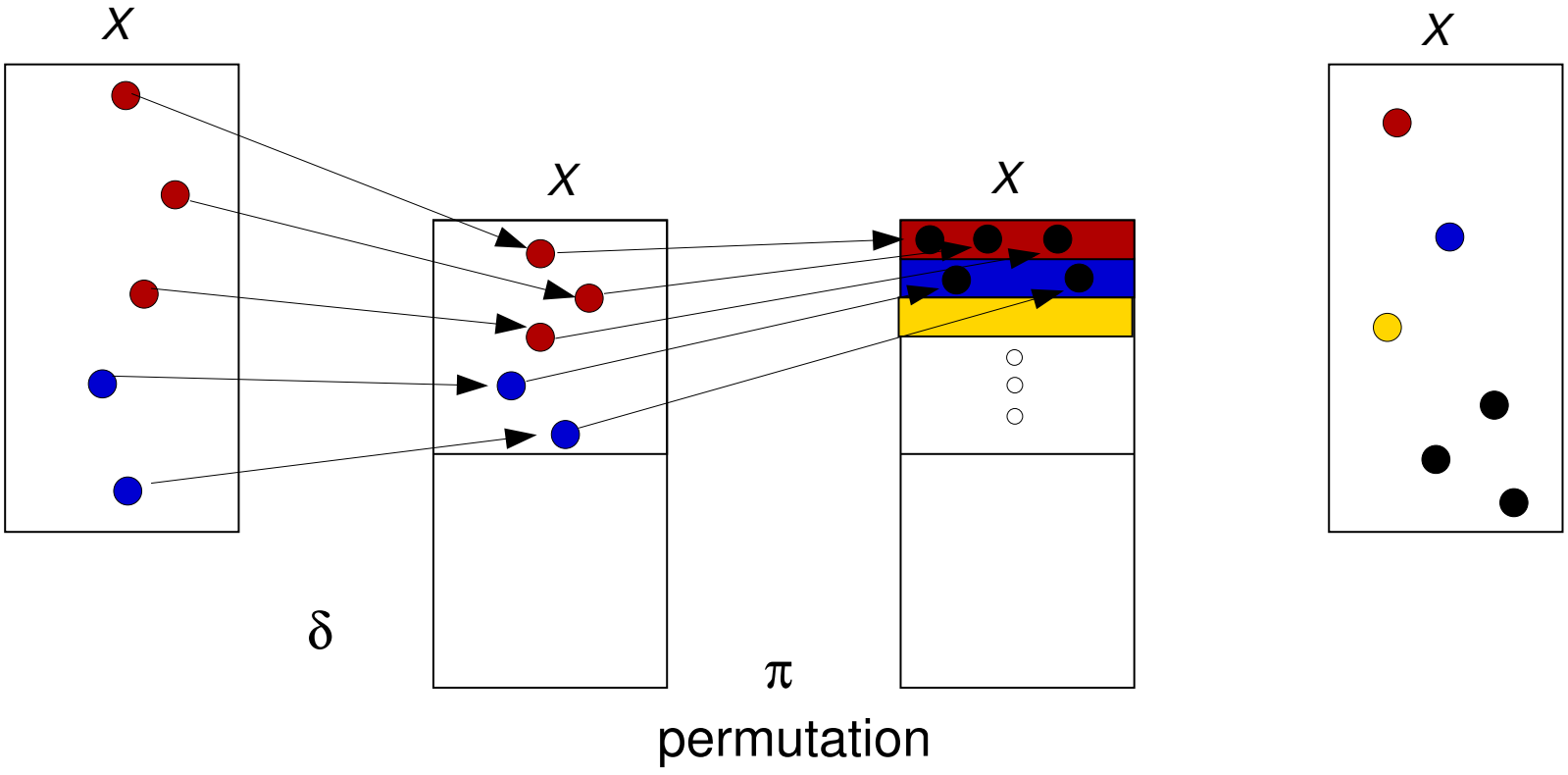

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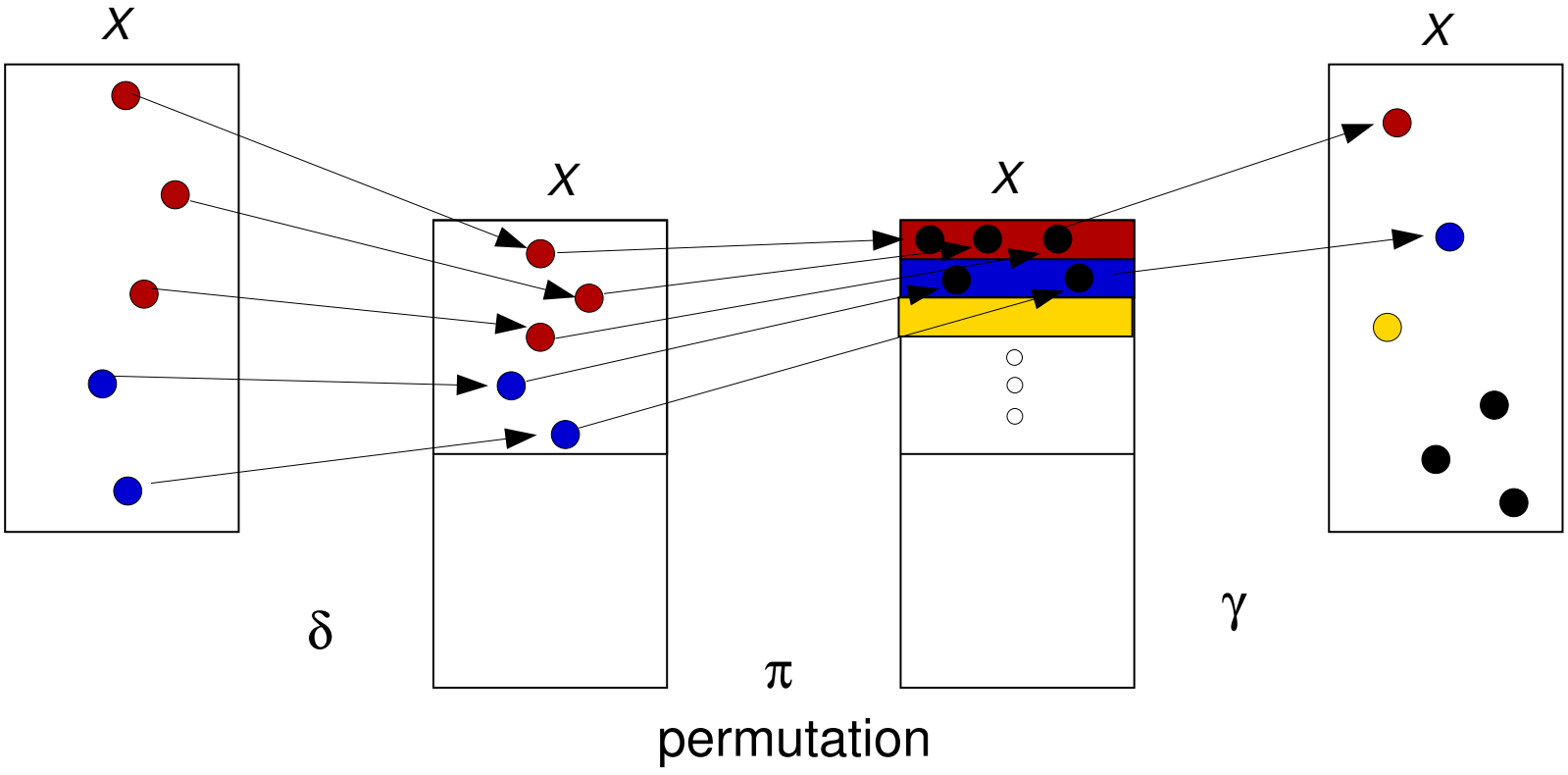

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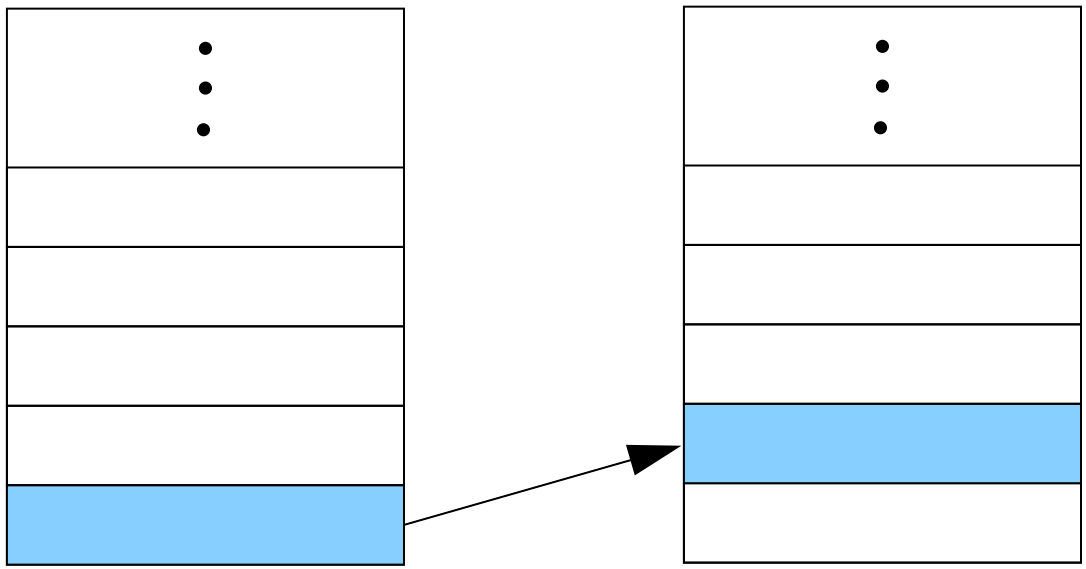

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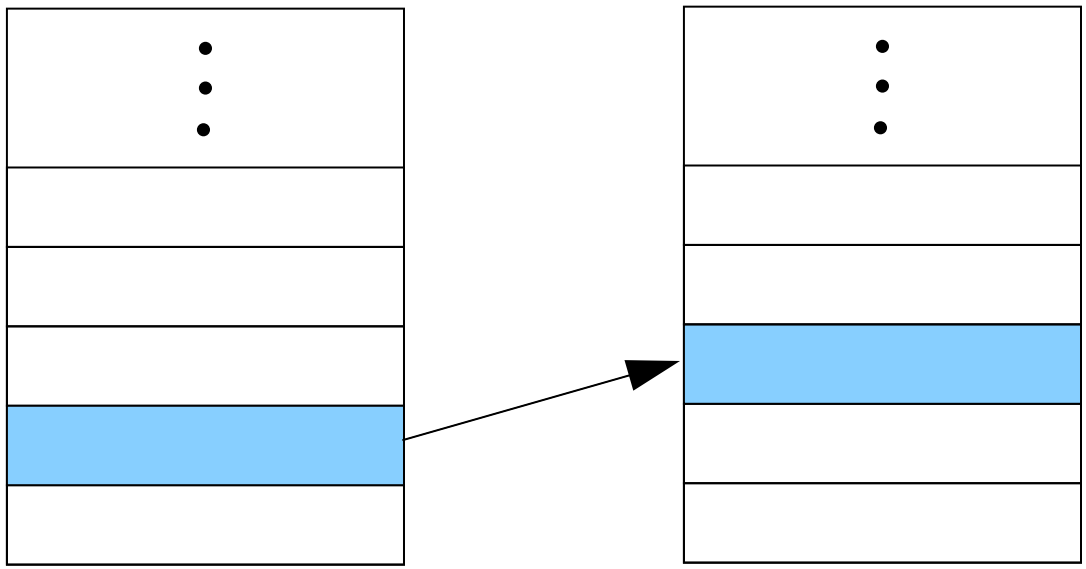

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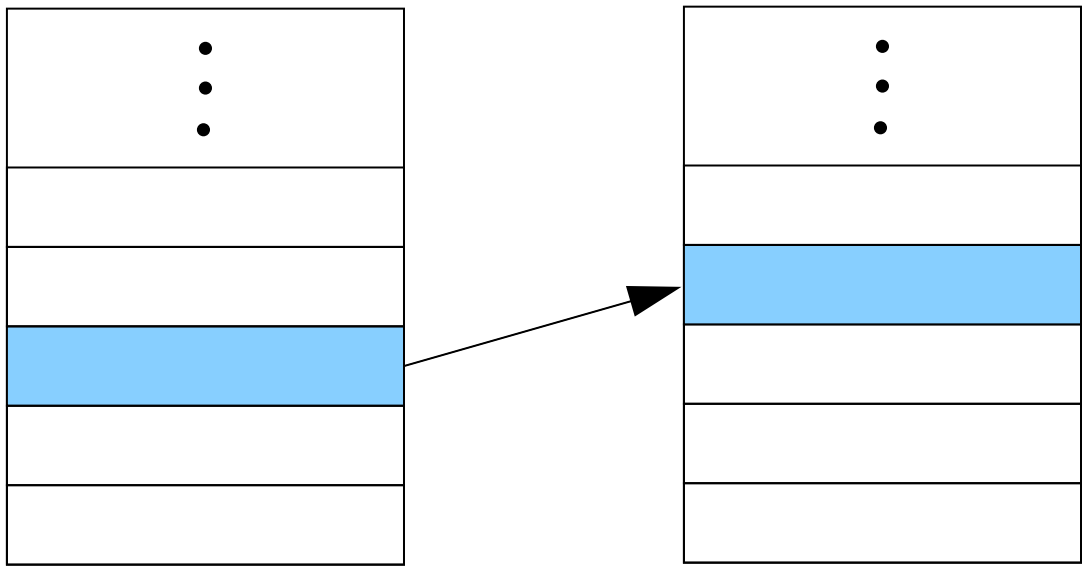

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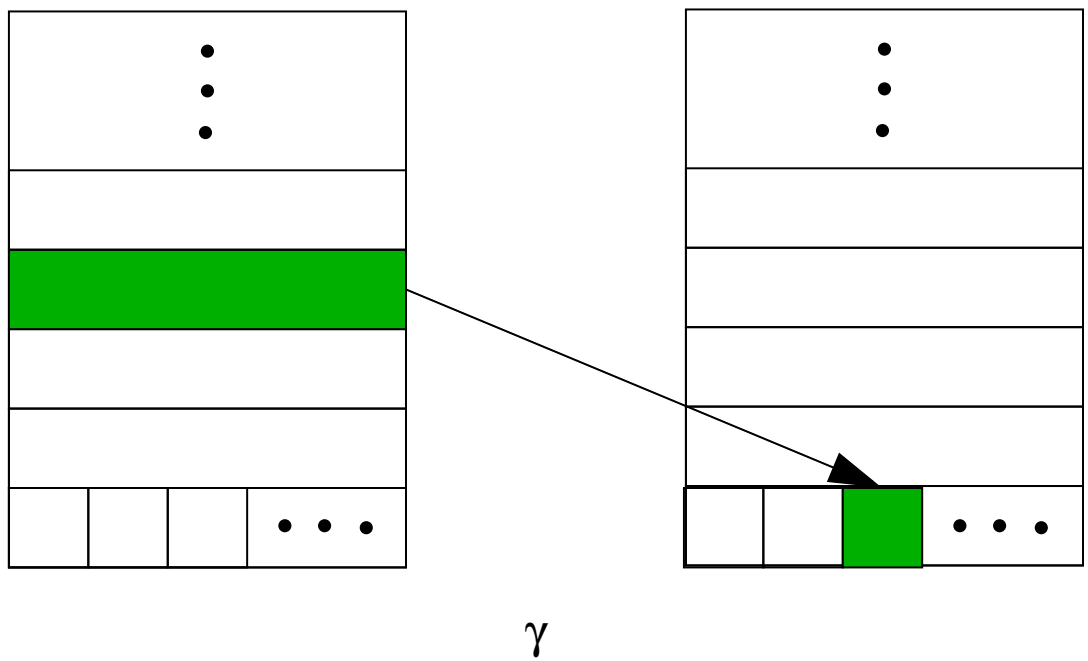

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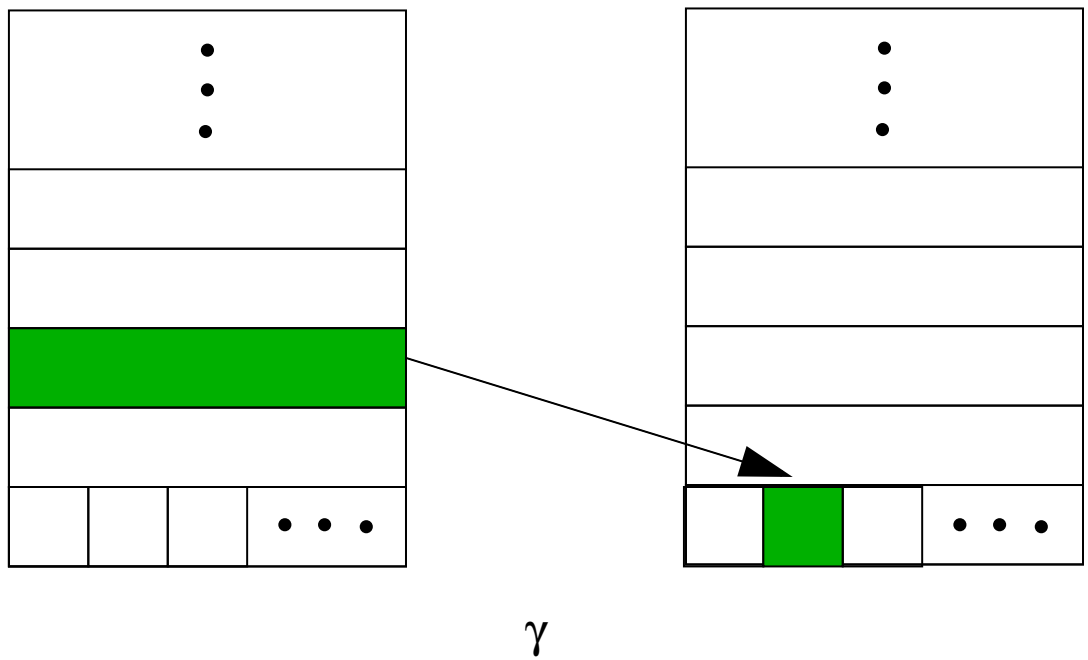

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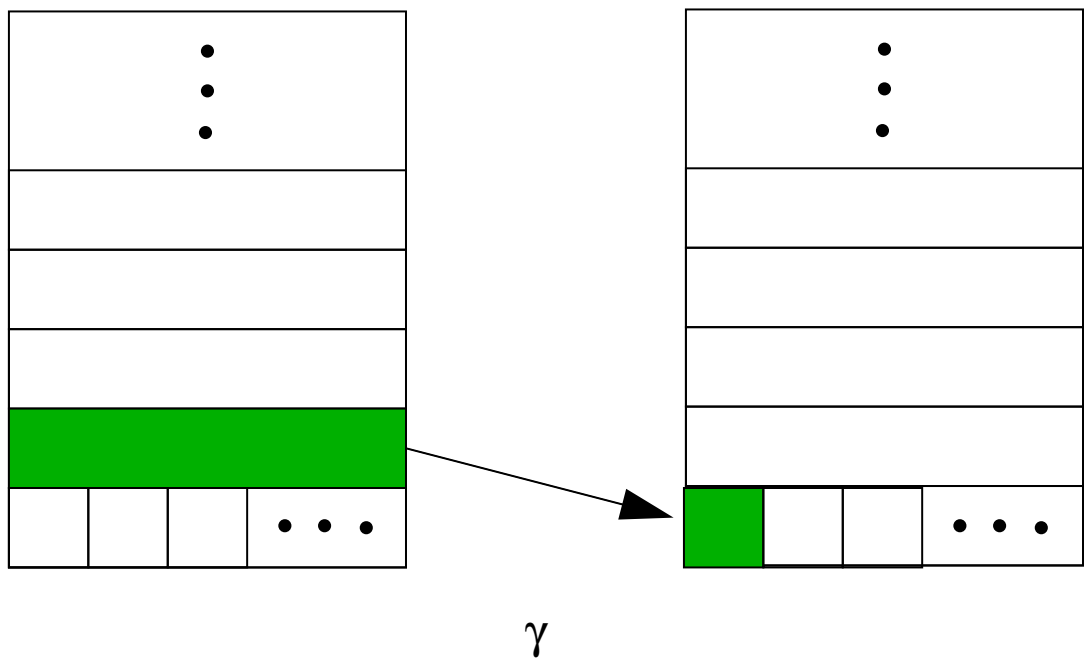

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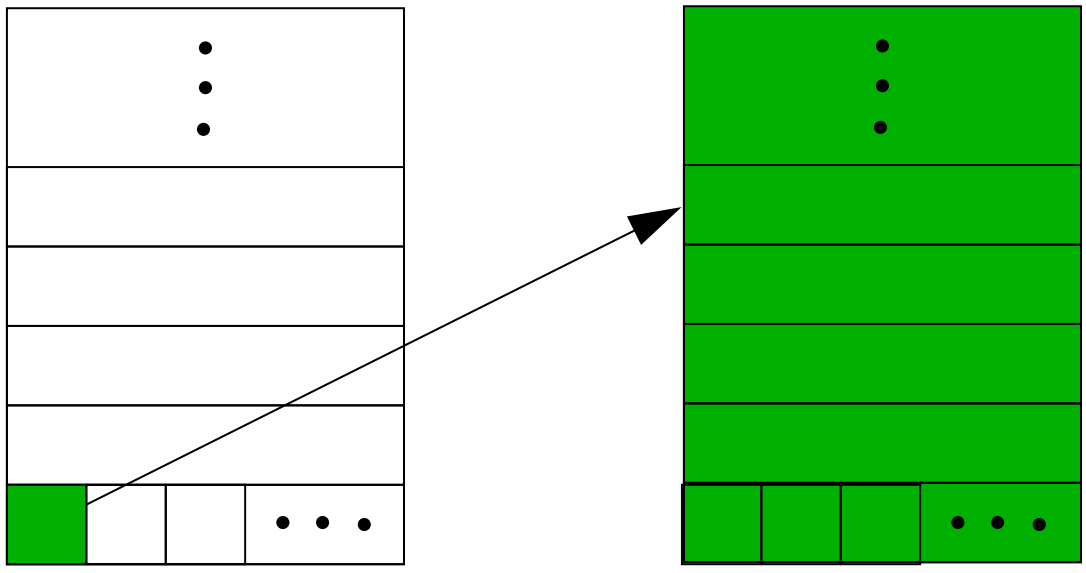

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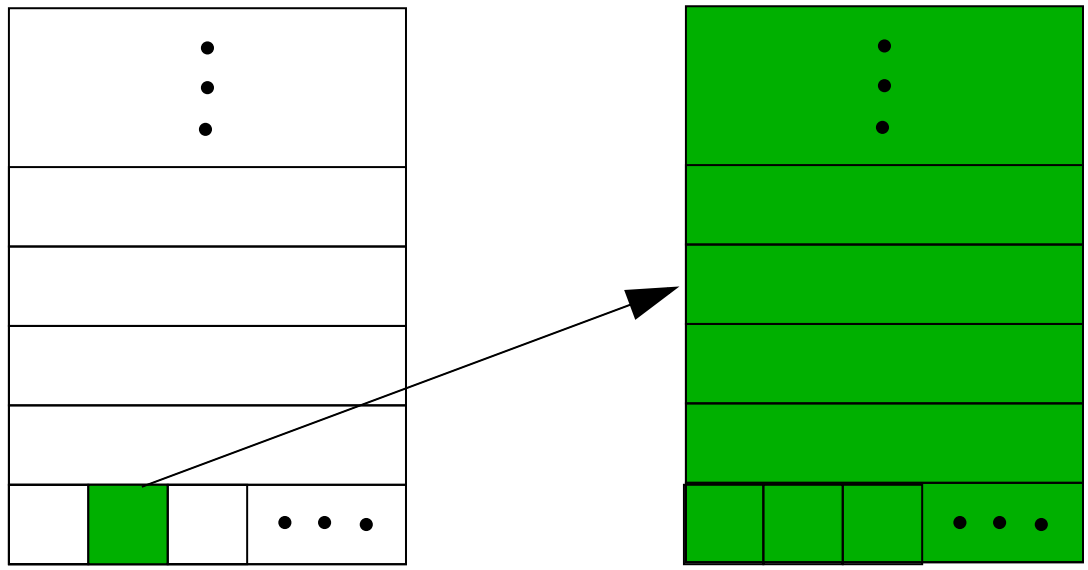

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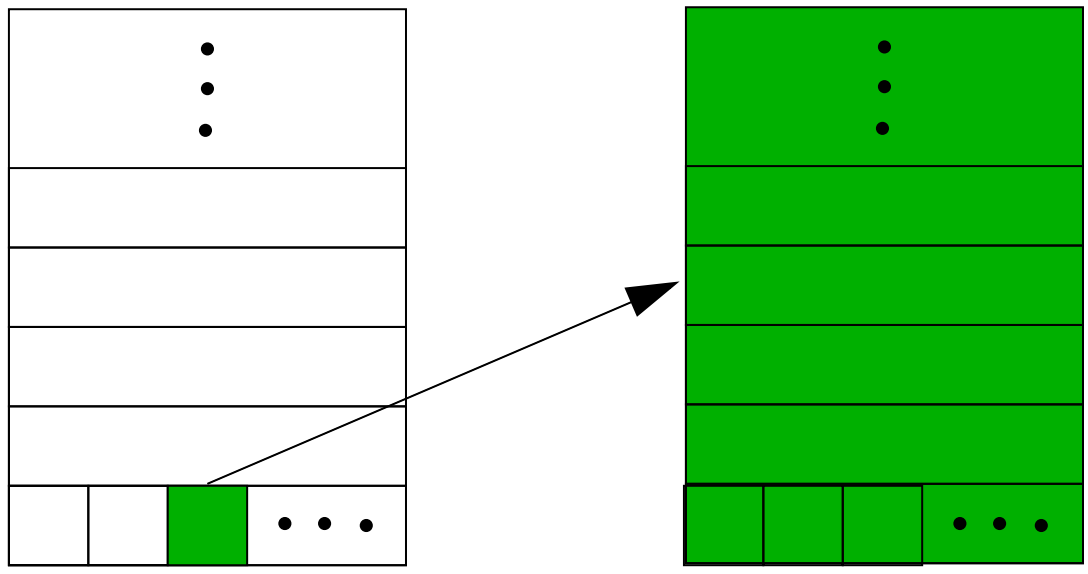

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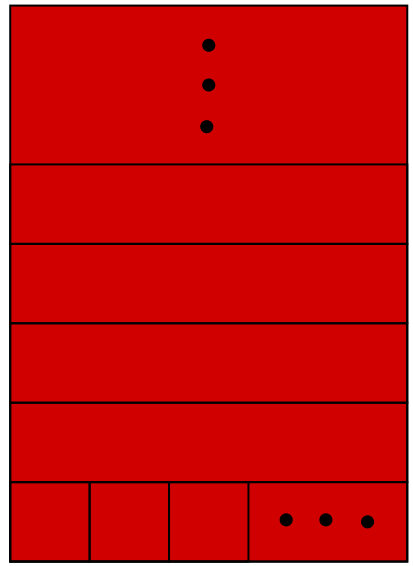

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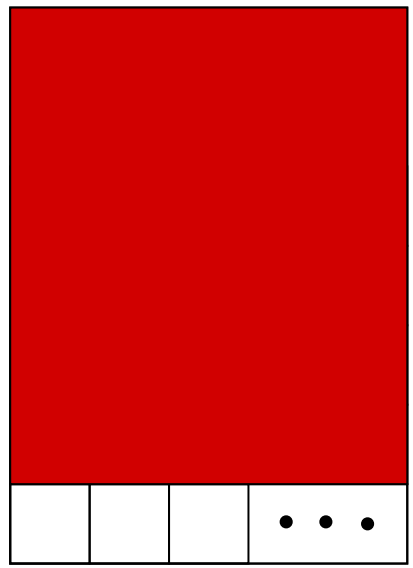

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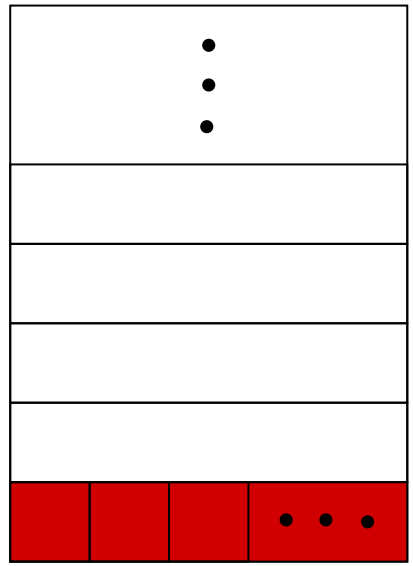

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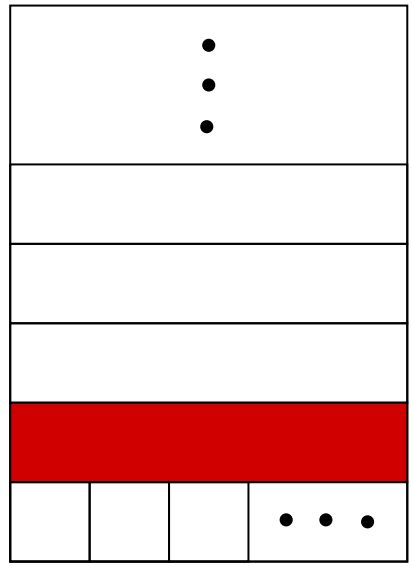

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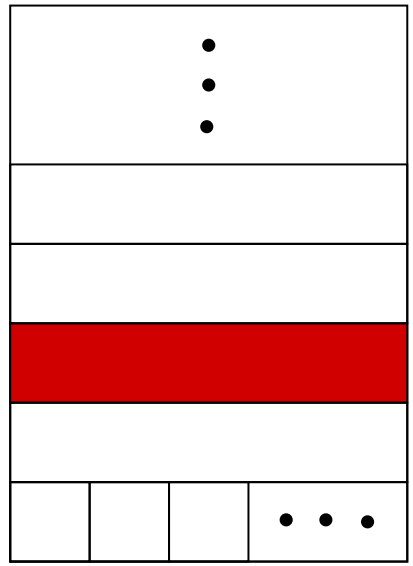

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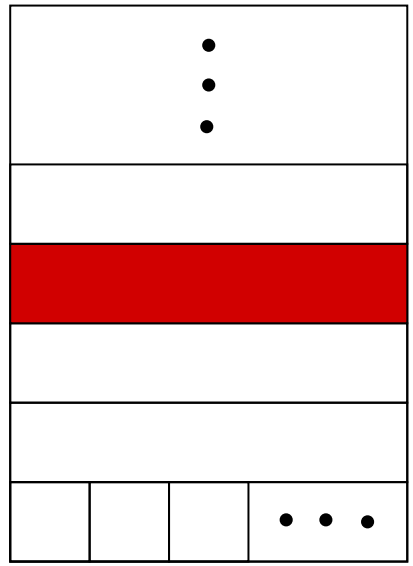

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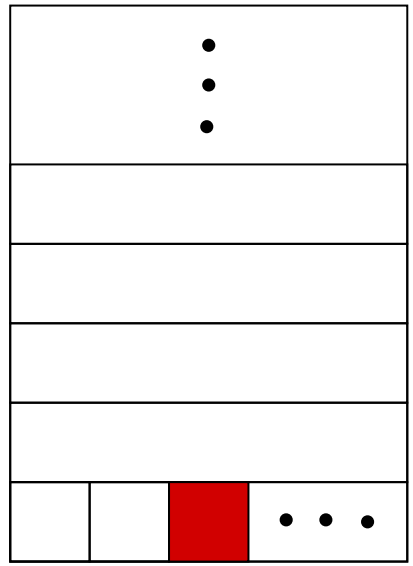

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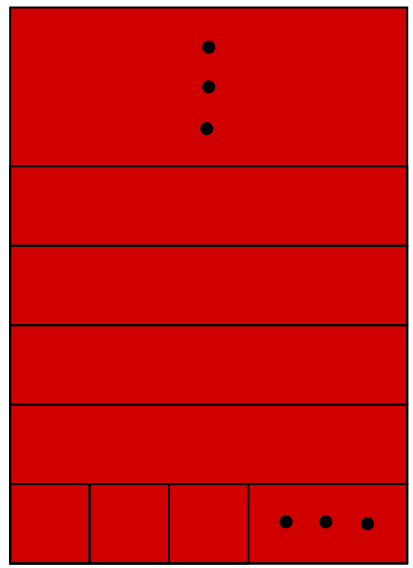

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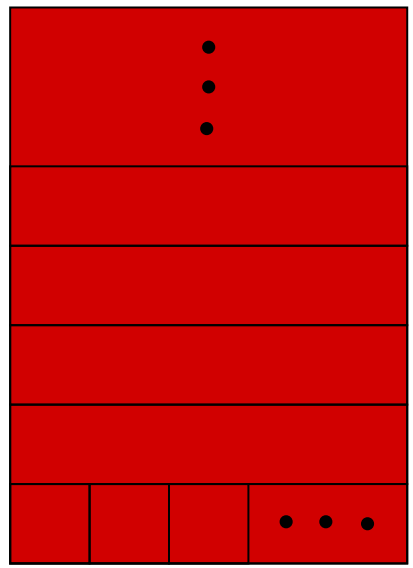

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We say that $T(X)$ has Sierpinski index 2.

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Every countable subset of $\mathrm{Trans}(X)$ *(*X *infinite) is contained in a 2-generated submonoid of* $\mathrm{Trans}(X)$*.*

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The relative rank of any submonoid of $\mathrm{Trans}(X)$ *is* 0, 1, 2 *or uncountable.*

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By Sierpinski's Theorem, $\tau_1, \tau_2, \ldots \in \langle \alpha, \beta \rangle$ for some α, β.

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Sierpinski/Galvin

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Theorem (Galvin 93)

Every countable subset of $\mathrm{Sym}(X)$ *(*X *infinite) is contained in a 2-generator subgroup of* $\mathrm{Sym}(X)$*. (*$\mathrm{Sym}(X)$ *has Sierpinski index* 2*.)*

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Continuous functions

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Problem

What is the Sierpinski index of $C((0,1))$*?*

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Theorem (Higgins, Howie, Mitchell, NR 03)

$\mathsf{rank}(\mathsf{Trans}(\mathbb{N}) : \mathsf{End}(\mathbb{N})) = 1.$

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Then $\gamma = \epsilon\delta \in \langle \mathrm{End}(\mathbb{N}), \delta \rangle$. □

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Classify, in order-theoretic terms, all linearly ordered sets L with finite relative ranks in $\mathrm{Trans}(L)$.

Partially ordered sets

Definition

Let P be a partially ordered set. For $x \in P$ write:

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(I) *Suppose c is finite. Then* $\mathrm{rank}(\mathrm{Trans}(P), \mathrm{End}(P)) \leq 2$ *if and only if there exists $x \in P$ such that at least one of $x^{\wedge}$ or $x^{\vee}$ is infinite.*

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(III) *If P is an antichain, then* $\mathrm{rank}(\mathrm{Trans}(P), \mathrm{End}(P)) = 0$:-)

Partially ordered sets

Corollary

$\mathrm{rank}(\mathrm{Trans}(P), \mathrm{End}(P)) \leq 2$ *provided any of the following hold:*

- P *has a smallest element;*
- P *has a largest element;*
- P *is a lattice.*
- *. . .*

A poset with relative rank 2 (I)

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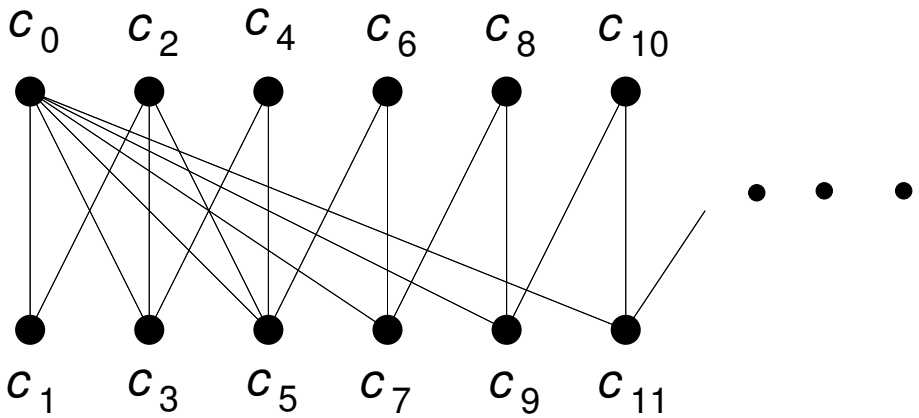
By induction $\pi = \mu^n$, a contradiction.

Poset with no monomorphisms



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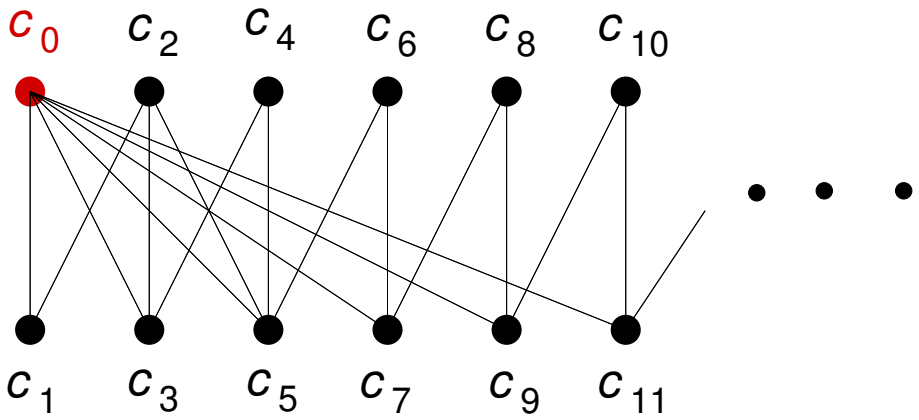

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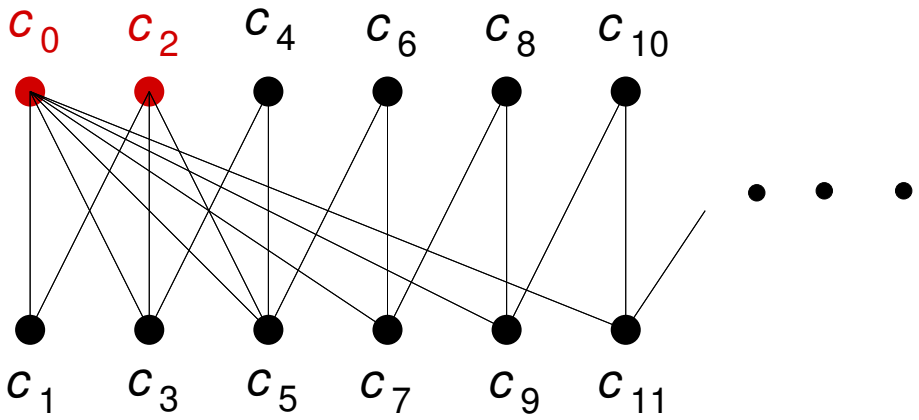

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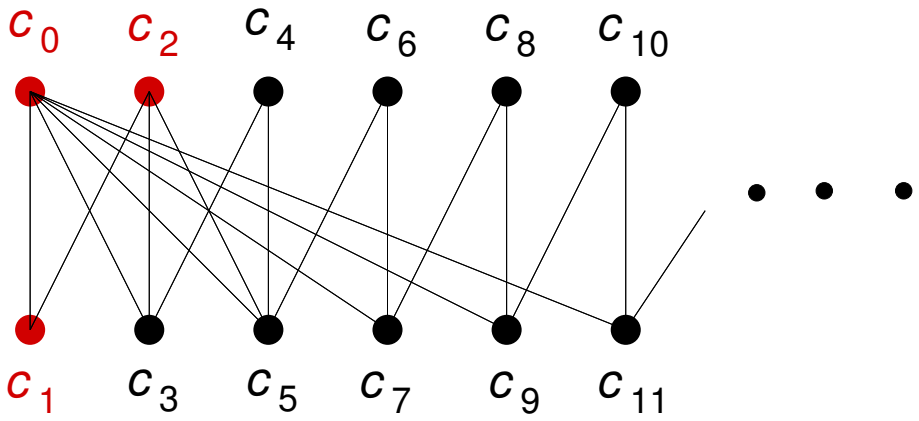

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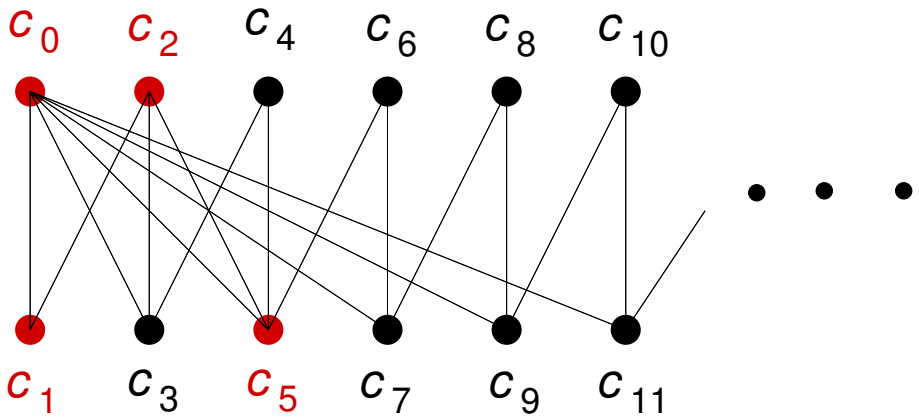

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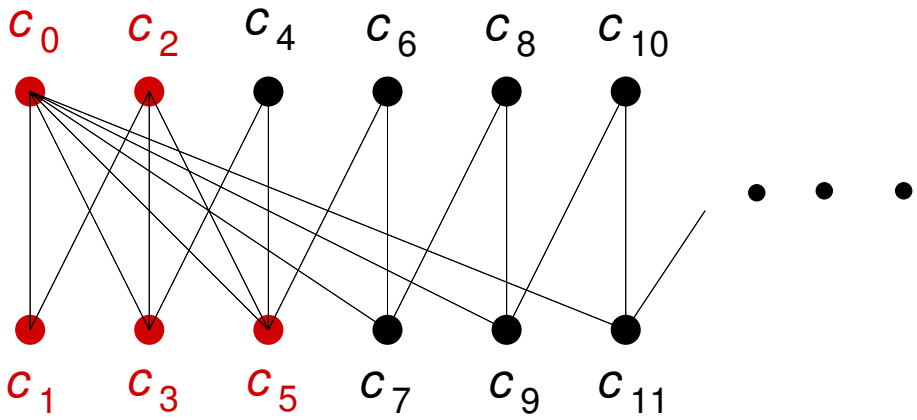

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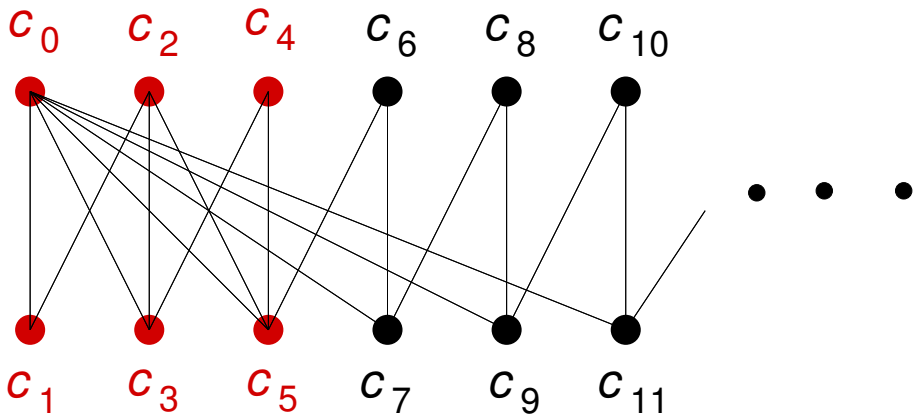

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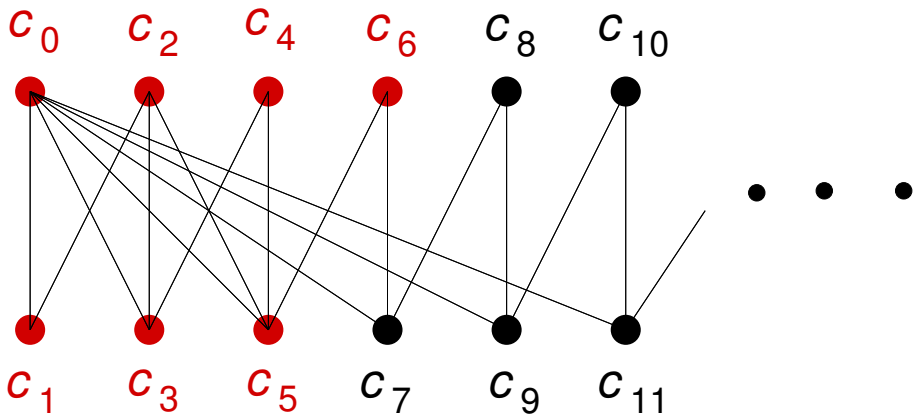

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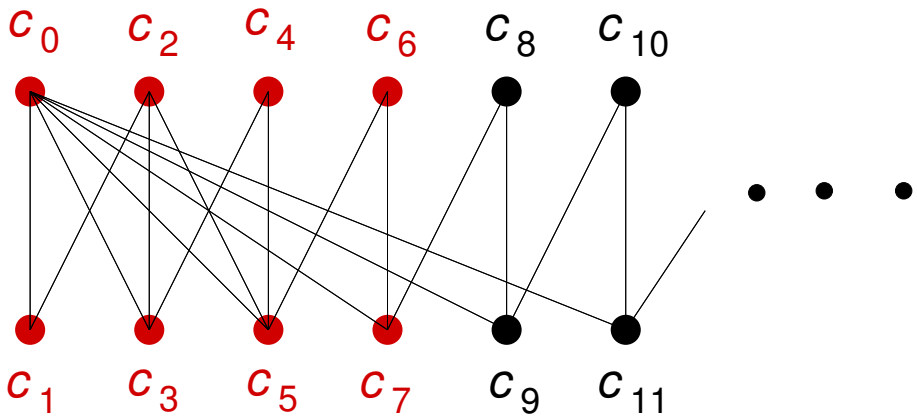

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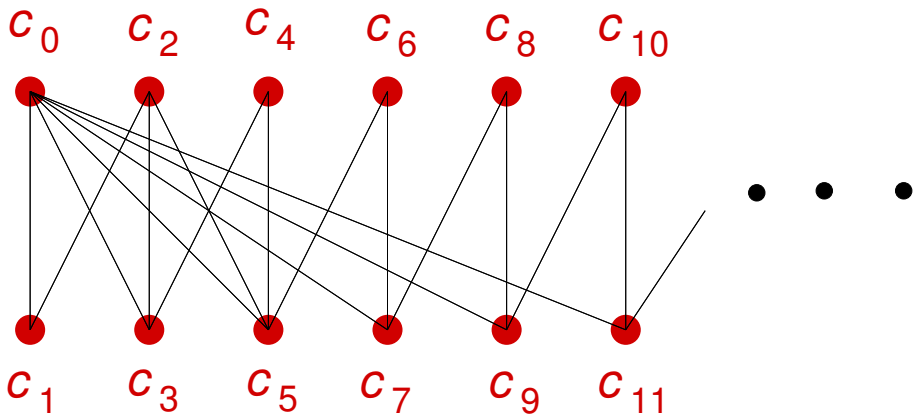

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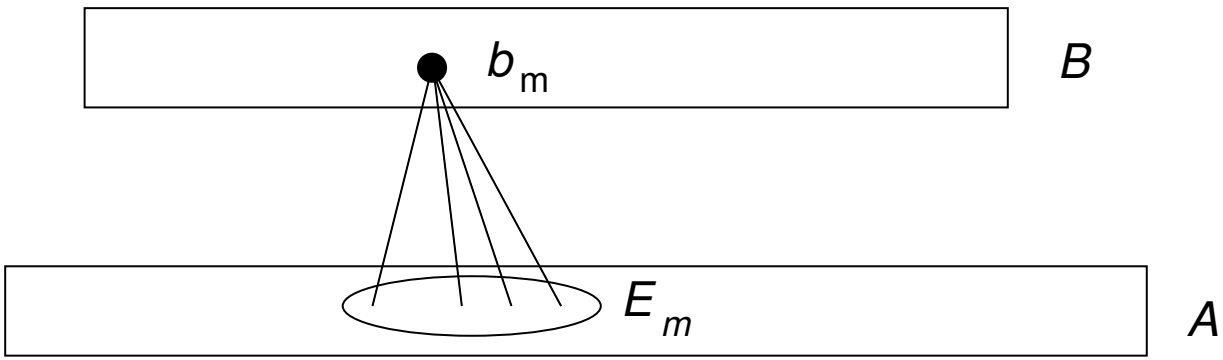
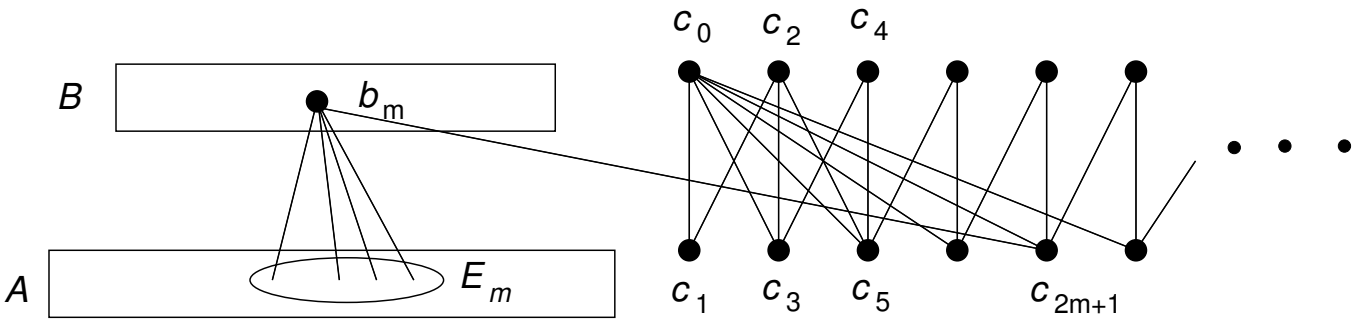
A partial order on $A \cup B$: $b_m > a_n \Leftrightarrow a_n \in E_m$.



A poset with relative rank 2 (II)



Proposition (Higgins, Mitchell, Morayne, NR 06)

The above poset P satisfies the conditions from the part (I) of the Theorem, but has no non-trivial mono- or epi-morphisms; consequently $\mathrm{rank}(\mathrm{Trans}(P) : \mathrm{End}(P)) = 2$.

The random graph R

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The random graph R is the unique countable graph with the following property:

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In fact, homogeneity and universality provide another definition of R.

Aut(R) and End(R): Similar?

Theorem (Truss 85)

Every countable group embeds into Aut(R).

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Theorem (Bonato, Delic, Dolinka)

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R is homomorphism homogeneous: If G and H are two finite induced subgraphs of R, and if $f : G \to H$ is a homomorphism, then there is an homomorphism $R \to R$ extending f.

Aut(R) and End(R): Or not similar?

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R is not characterised by the properties of universality and homomorphism homogeneity.

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To every ideal I of a monoid M there corresponds the Rees congruence $\Phi_I \cup \Delta_M$ on M; but, not every congruence is a Rees congruence.

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Question

How many non-Rees congruences does End(R) have?

Random graph: Questions

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Does Aut(R) have a finite Sierpinski index?

Random graph: Questions

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Facts

$\mathrm{rank}(\mathrm{Sym}(R) : \mathrm{Aut}(R)) = 1$, $\mathrm{rank}(\mathrm{Trans}(R) : \mathrm{Aut}(R)) = 2$,
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Question

$\mathrm{rank}(\mathrm{End}(R) : \mathrm{Aut}(R)) = ?$