

Endomorphism Monoids of Combinatorial Structures

Nik Ruskuc

`nik@mcs.st-and.ac.uk`

School of Mathematics and Statistics, University of St Andrews

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University
of
St Andrews

Relational structures

Definition

- ▶ A **relational structure** $\mathcal{X} = (X; R_i (i \in I))$ consists of a set X and a collection of relations defined on X .
- ▶ Each relation R_i has an **arity** r_i , meaning $R_i \subseteq X^{r_i}$.
- ▶ The sequence $(r_i (i \in I))$ is called the **signature** of \mathcal{X} .

Combinatorial structures as relational structures

Different combinatorial structures can be distinguished by their signatures and special properties required from their relations.

Examples

- ▶ A (simple, undirected) graph is a relational structure with signature (2) , such that its only binary relation is symmetric and irreflexive.
- ▶ A general structure of signature (2) is a digraph.
- ▶ Posets: signature (2) ; properties R, AS, T.
- ▶ Permutations: signature $(2,2)$; properties: two linear orders.

Morphisms

Definition

Let $\mathcal{X} = (X; R_i (i \in I))$ be a relational structure. An **endomorphism** is a mapping $\theta : X \rightarrow X$ which respects all the relations R_i , i.e.

$$(x_1, \dots, x_k) \in R_i \Rightarrow (x_1\theta, \dots, x_k\theta) \in R_i.$$

An **automorphism** is an invertible endomorphism.

Remark

For posets and graphs the above becomes:

$$x \leq y \Rightarrow x\theta \leq y\theta,$$

$$x \sim y \Rightarrow x\theta \sim y\theta.$$

Warning: A bijective endomorphism need not be invertible. Take $V = \mathbb{Z}$, $E = \{(i-1, i) : i \leq 0\}$, and $f : x \mapsto x-1$.

Morphisms

$\text{End}(\mathcal{X})$ = the endomorphism monoid of \mathcal{X}

$\text{Aut}(\mathcal{X})$ = the automorphism group of \mathcal{X}

General Problem

For a given \mathcal{X} , how are \mathcal{X} , $\text{End}(\mathcal{X})$ and $\text{Aut}(\mathcal{X})$, and their properties, related?

Trans(X) and Sym(X)

Let $\mathcal{E} = \mathcal{E}(X)$ be a trivial relational structure on X .

$\text{Trans}(X) = \text{End}(\mathcal{E})$ – the **full transformation monoid** on X

$\text{Sym}(X) = \text{Aut}(\mathcal{E})$ – the **symmetric group** on X

Facts

- ▶ $|\text{Trans}(n)| = n^n$, $|\text{Sym}(n)| = n!$
- ▶ (X infinite) $|\text{Trans}(X)| = |\text{Sym}(X)| = 2^{|X|}$

Facts

- ▶ Every monoid embeds into some $\text{Trans}(X)$.
- ▶ Every group embeds into some $\text{Sym}(X)$.

Finite chains

C_n : $1 < 2 < \dots < n$

Facts

- ▶ $\text{Aut}(C_n) = \{\text{id}\}$
- ▶ $|\text{End}(C_n)| = \binom{2n-1}{n-1}$

Graphs, posets, permutations

Theorem

- ▶ *Every group is isomorphic to $\text{Aut}(G)$ for some graph G .
(Frucht, 1938)*
- ▶ *Every monoid is isomorphic to $\text{End}(G)$ for some graph G .
(Hedrlín, Pultr, 1965)*
- ▶ *Every group is isomorphic to $\text{Aut}(P)$ for some poset P .
(Birkhoff, 1946)*
- ▶ *Not every monoid is isomorphic to $\text{End}(P)$ of some poset P .*

Problem

Describe the monoids $\text{End}(P)$ where P is a permutation.

Rank

Definition

The **rank** of a monoid M is the smallest number of elements needed to generate M ; notation: $\text{rank}(M)$.

Facts

- ▶ $\text{rank}(\text{Sym}(n)) = 2$.
- ▶ $\text{rank}(\text{Trans}(n)) = 3$.
- ▶ $\text{rank}(\text{End}(C_n)) = n$.

Question

What is $\text{rank}(B_n)$, where B_n is the monoid of all binary relations on $\{1, \dots, n\}$?

Relative ranks

For uncountable monoids, the notion of rank is vacuous ($\text{rank}(M) = |M|$).

Definition

Let M be a monoid, and let N be a submonoid of M . The **relative rank** of M modulo N (denoted $\text{rank}(M : N)$) is the smallest size of a set A such that $M = \langle N \cup A \rangle$.

Example

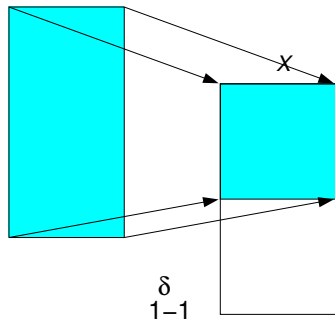
$\text{rank}(\text{Trans}(n) : \text{Sym}(n)) = 1$.

Example: $\text{rank}(\text{Trans}(X) : \text{Sym}(X))$

Proposition (Higgins, Howie, NR 98)

For X infinite, $\text{rank}(\text{Trans}(X) : \text{Sym}(X)) = 2$.

Proof. (\leq)



Some more relative ranks

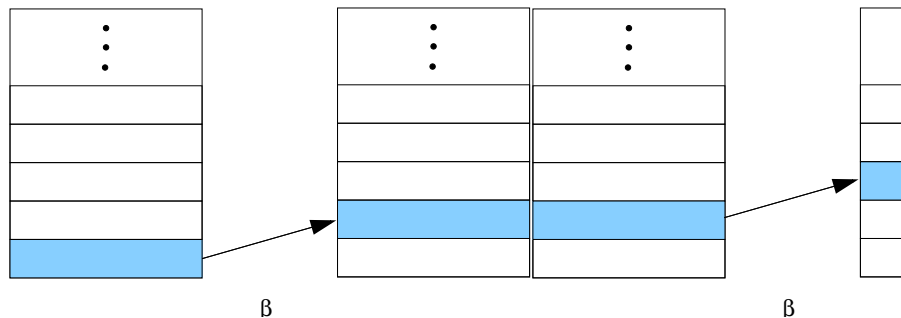
- ▶ $\text{rank}(\text{Trans}(X) : \langle E(\text{Trans}(X)) \rangle) = 2.$
- ▶ $\text{rank}(\text{Trans}(X) : \text{Inj}(X)) = 1.$
- ▶ $\text{rank}(\text{Trans}(X) : \text{Surj}(X)) = 1.$

Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset $\{\theta_1, \theta_2, \dots\}$ of $\text{Trans}(X)$ (X infinite) is contained in a 2-generated submonoid of $\text{Trans}(X)$.

Proof. [Banach 35]



We say that $T(X)$ has **Sierpinski index 2**.

Sierpinski's Theorem: Application to relative ranks

Theorem (Sierpinski 35)

Every countable subset of $\text{Trans}(X)$ (X infinite) is contained in a 2-generated submonoid of $\text{Trans}(X)$.

Corollary

The relative rank of any submonoid of $\text{Trans}(X)$ is 0, 1, 2 or uncountable.

Proof

Proof.

Suppose $\text{rank}(\text{Trans}(X) : S) = \aleph_0$, with $\text{Trans}(X) = \langle S, \tau_1, \tau_2, \dots \rangle$.
By Sierpinski's Theorem, $\tau_1, \tau_2, \dots \in \langle \alpha, \beta \rangle$ for some α, β . But
then $\text{Trans}(X) = \langle S, \alpha, \beta \rangle$. □

Sierpinski/Galvin

Theorem (Galvin 93)

Every countable subset of $\text{Sym}(X)$ (X infinite) is contained in a 2-generator subgroup of $\text{Sym}(X)$. ($\text{Sym}(X)$ has Sierpinski index 2.)

Corollary

The relative rank of any subgroup G of $\text{Sym}(X)$ is 0, 1 or uncountable.

Continuous functions

Theorem (Sierpinski 34; Mitchell, Peresse, Quick 07)

$C([0, 1])$ has Sierpinski index 2.

Problem

What is the Sierpinski index of $C((0, 1))$?

Linearly ordered set \mathbb{N}

Theorem (Higgins, Howie, Mitchell, NR 03)

$$\text{rank}(\text{Trans}(\mathbb{N}) : \text{End}(\mathbb{N})) = 1.$$

Proof.

Let $\delta : \mathbb{N} \rightarrow \mathbb{N}$ be any onto, infinite-to-one, mapping.

Let $\gamma \in \text{Trans}(\mathbb{N})$ be arbitrary.

Define $\epsilon \in \text{End}(\mathbb{N})$ inductively by:

$$k\epsilon \in (k\gamma)\delta^{-1} \text{ and } k\epsilon > (k-1)\epsilon.$$

Then $\gamma = \epsilon\delta \in \langle \text{End}(\mathbb{N}), \delta \rangle$. □

Linearly ordered sets

Higgins, Mitchell, NR 03

Theorem

If L is a countable linearly ordered set then

$$\text{rank}(\text{Trans}(L) : \text{End}(L)) = 1.$$

Theorem

If L is a well-ordered set then $\text{rank}(\text{Trans}(L) : \text{End}(L)) = 1$.

Example

$\text{rank}(\text{Trans}(\mathbb{R}) : \text{End}(\mathbb{R}))$ is uncountable.

Question

Does there exist an infinite linearly ordered set L such that $\text{rank}(\text{Trans}(L) : \text{End}(L)) = 2$?

Question

Classify, in order-theoretic terms, all linearly ordered sets L with finite relative ranks in $\text{Trans}(L)$.

Partially ordered sets

Definition

Let P be a partially ordered set. For $x \in P$ write:

$$x^\wedge = \{p \in P : x \leq p\}, \quad x^\vee = \{p \in P : p \geq x\}$$

Theorem (Higgins, Mitchell, Morayne, NR 06)

Let P be a countably infinite poset, and let c be the number of connected components of P .

- (I) Suppose c is finite. Then $\text{rank}(\text{Trans}(P), \text{End}(P)) \leq 2$ if and only if there exists $x \in P$ such that at least one of x^\wedge or x^\vee is infinite.
- (II) If c is infinite and P is not an antichain, then $\text{rank}(\text{Trans}(P), \text{End}(P)) = 1$.
- (III) If P is an antichain, then $\text{rank}(\text{Trans}(P), \text{End}(P)) = 0$:-)

Partially ordered sets

Corollary

$\text{rank}(\text{Trans}(P), \text{End}(P)) \leq 2$ provided any of the following hold:

- ▶ P has a smallest element;
- ▶ P has a largest element;
- ▶ P is a lattice.
- ▶ ...

A poset with relative rank 2 (I)

Higgins, Mitchell, Morayne, NR 06

Lemma

If P is a poset with no non-trivial mono- or epi-morphisms then $\text{rank}(\text{Trans}(P), \text{End}(P)) \geq 2$.

Proof. Suppose $\text{Trans}(P) = \langle \text{End}(P), \mu \rangle$.

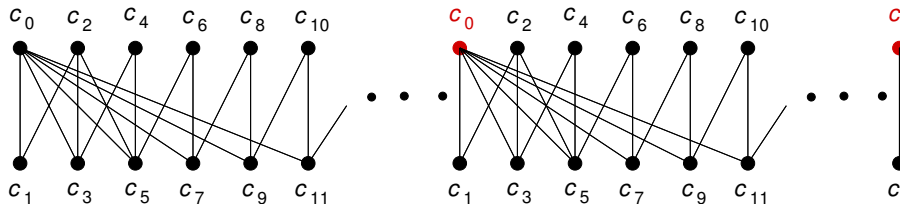
Let $\pi \in \text{Sym}(P)$; write $\pi = \gamma_1 \gamma_2 \dots \gamma_n$, $\gamma_i \in \text{End}(P) \cup \{\mu\}$.

γ_1 is injective; γ_n is surjective.

Hence $\gamma_1 = \gamma_n = \mu \in \text{Sym}(P)$.

By induction $\pi = \mu^n$, a contradiction.

Poset with no monomorphisms



Poset with no epimorphisms

Let $A = \{a_1, a_2, \dots\}$ be a countably infinite set.

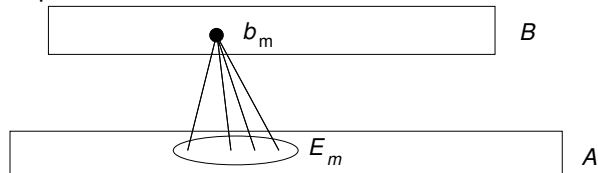
Let \mathcal{E} be the set of finite subsets E of A satisfying:

$a_n \in E \Rightarrow 2 \leq |E| \leq n + 1$.

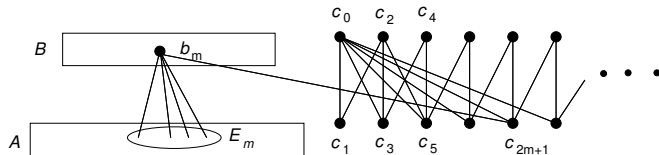
Enumerate \mathcal{E} : E_1, E_2, \dots

Let $B = \{b_1, b_2, \dots\}$.

A partial order on $A \cup B$: $b_m > a_n \Leftrightarrow a_n \in E_m$.



A poset with relative rank 2 (II)



Proposition (Higgins, Mitchell, Morayne, NR 06)

The above poset P satisfies the conditions from the part (I) of the Theorem, but has no non-trivial mono- or epi-morphisms; consequently $\text{rank}(\text{Trans}(P) : \text{End}(P)) = 2$.

The random graph R

Definition

The **random graph R** is the unique countable graph with the following property: For any two disjoint finite sets of vertices U , V , there exists a vertex v adjacent to all vertices in U and not adjacent to any in V .

Fact

R is homogeneous: *If G and H are two finite induced subgraphs of R , and if $f : G \rightarrow H$ is an isomorphism, then there is an automorphism $R \rightarrow R$ extending f .*

Fact

R is universal: *R contains a copy of every countable graph.*

In fact, homogeneity and universality provide another definition of R .

Aut(R) and End(R): Similar?

Theorem (Truss 85)

Every countable group embeds into Aut(R).

Theorem (Bonato, Delic, Dolinka)

Every countable monoid embeds into End(R).

Theorem

R is homomorphism homogeneous: If G and H are two finite induced subgraphs of R , and if $f : G \rightarrow H$ is a homomorphism, then there is an homomorphism $R \rightarrow R$ extending f .

Aut(R) and End(R): Or not similar?

Fact (Cameron, Nešetřil)

R is *not* characterised by the properties of universality and homomorphism homogeneity.

Theorem (Truss 85)

Aut(R) is simple.

Theorem (Delic, Dolinka 04)

End(R) has uncountably many ideals.

Remark

To every ideal I of a monoid M there corresponds the Rees congruence $\Phi_I \cup \Delta_M$ on M ; but, not every congruence is a Rees congruence.

Question

How many non-Rees congruences does End(R) have?

Random graph: Questions

Question

Does $\text{Aut}(R)$ have a finite Sierpinski index?

Question

Does $\text{End}(R)$ have a finite Sierpinski index?

Facts

$\text{rank}(\text{Sym}(R) : \text{Aut}(R)) = 1$, $\text{rank}(\text{Trans}(R) : \text{Aut}(R)) = 2$,
 $\text{rank}(\text{Trans}(R) : \text{End}(R)) = 1$.

Question

$\text{rank}(\text{End}(R) : \text{Aut}(R)) = ?$