Some interesting properties of direct products of groups, semigroups and other algebraic structures

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University of St Andrews

Preview: $1 + 1 = 2$

...for I cannot satisfy myself that, when one is added to one, the one to which the addition is made becomes two, or that the two units added together make two by reason of the addition. I cannot understand how, when separated from the other, each of them was one and not two, and now, when they are brought together, the mere juxtaposition or meeting of them should be the cause of their becoming two: neither can I understand how the division of one is the way to make two; for then a different cause would produce the same effect,–as in the former instance the addition and juxtaposition of one to one was the cause of two, in this the separation and subtraction of one from the other would be the cause. Nor am I any longer satisfied that I understand the reason why one or anything else is either generated or destroyed or is at all, but I have in my mind some confused notion of a new method ...

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Instead of an Introduction: $d(A_n)$, $d(S_n)$

Definition

The rank of an algebraic structure is the smallest number of generators needed to generate A; it is denoted d(A).

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Facts

$d(A_5^{19}) = 2$, $d(A_5^{20}) = 3$.

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$$A \xleftarrow{\pi_A \,:\, (a,b) \longrightarrow a} A \times B \xrightarrow{\pi_B \,:\, (a,b) \longrightarrow b} B$$

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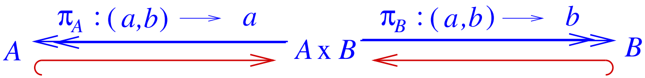

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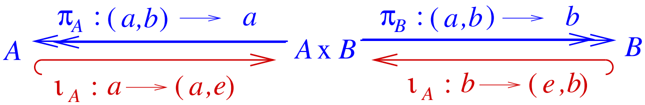

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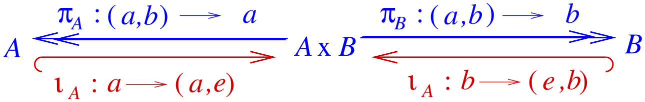

Provided e is an idempotent

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Remark

This works for monoids.

Growth of Direct Powers

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For any monoids M, N we have

$$\max(d(M), d(N)) \leq d(M \times N) \leq d(M) + d(N).$$

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Growth Sequences: Finite Groups

J. Wiegold (with J.S. Wilson, D. Meier, A.G.R. Stewart, A. Efranian), 1974–1995.

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Does there exist an infinite simple group G such that $d(G^n) = d(G) + 1$?

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If a perfect group G has no finite images, is $\mathbf{d}(G)$ necessarily eventually constant?

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In the non-monoid case, is $\mathbf{d}(S)$ eventually exponential?

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Does such a monoid exist?

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Example

The monoid of all recursive functions $\mathbb{N} \to \mathbb{N}$ is finitely generated, and also has the above property.

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Theorem

Let S, T be semigroups, with S infinite, T finite. $S \times T$ is finitely generated if and only if

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Question

Will $S \times T$ be finitely presented provided S and T are finitely presented and $S \times T$ is finitely generated?

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Two words u, v over A are equal in S if and only if there is a sequences of applications of relations from R (a deduction) which transforms u into v.

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Remark

The above definition of stability is not constructive.

Stability and Finite Presentability

Theorem (EF Robertson, NR, J Wiegold)

Let S, T be two infinite semigroups. $S \times T$ is finitely presented if and only if

(i) *S and T are (finitely presented) and stable; and*

(ii) *neither S nor T contain indecomposable elements.*

Theorem (EF Robertson, NR, J Wiegold)

Let S be an infinite semigroup, and let T be a finite semigroup. $S \times T$ is finitely presented if and only if

(i) *S is finitely presented; and*

(ii) *T is stable and contains no indecomposable elements.*

Finite Presentability: 'Good Classes'

Corollary

Suppose that S and T belong to any of the following classes: monoids (including groups), regular semigroups (including inverse semigroups), surjective commutative semigroups (...) Then $S \times T$ is finitely presented if and only if S and T are finitely presented.

Some Non-Finitely-Presented Examples

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The four element semigroup

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is a non-stable semigroup of minimal size. Hence, for example, $S \times \mathbb{Z}$ is finitely generated but not finitely presented.

A Digression: A Strange Union

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Note that $S = U \dot{\cup} V$, where $U = \{a\} \leq S$, $V = \{b, c, 0\} \leq S$.

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Residual Finiteness: General, Nice, Boring Theorem?

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If $e \in A$ is an idempotent, then $B \cong \{e\} \times B \leq A \times B$.

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R. Gray, NR

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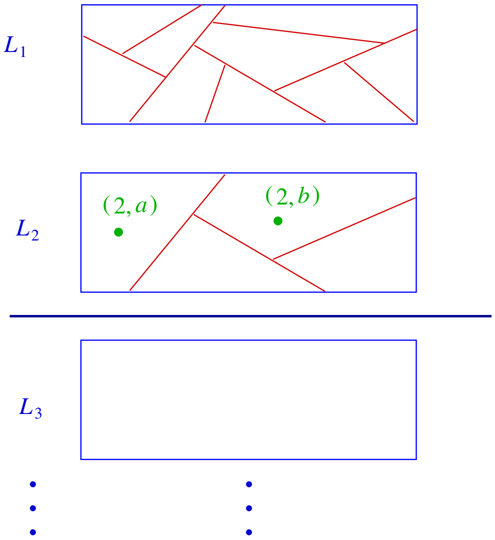
The equivalence relation λ with classes L_1, L_2, $L_3 \cup L_4 \cup \ldots$ is a congruence with finitely many classes.
Intersect ρ and λ to obtain a congruence $\sigma = \rho \cap \lambda$ which has finitely many classes, respects levels 1 and 2, and separates $(2, a)$ and $(2, b)$.

Residual Finiteness: Levels of $\mathbb{N} \times S$



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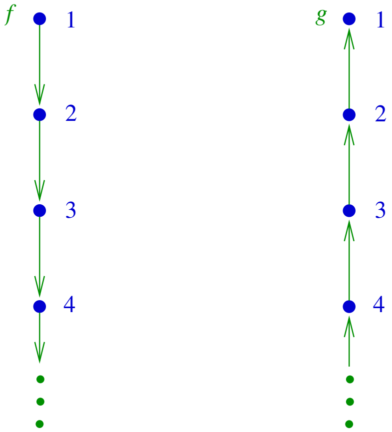
Let S and T be semigroups. $S \times T$ is residually finite if and only if S and T are residually finite.

Proof
A semigroup either contains an idempotent or a copy of $\mathbb{N}$.

Residual Finiteness: Unary Algebras

Example

Consider two unary operations on $\mathbb{N}$:



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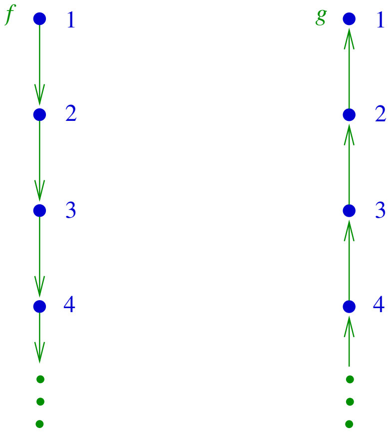

Let $A = (\mathbb{N}, f)$, $B = (\mathbb{N}, g)$.

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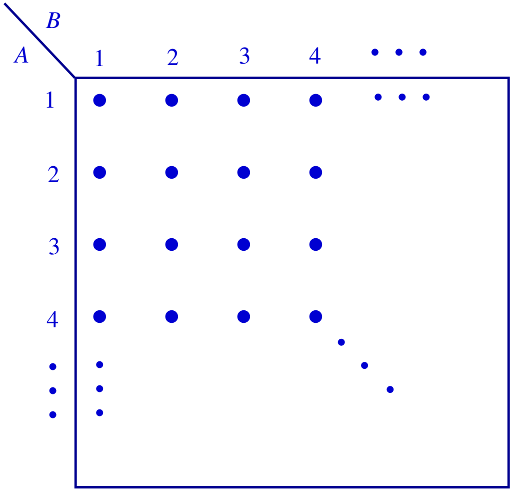

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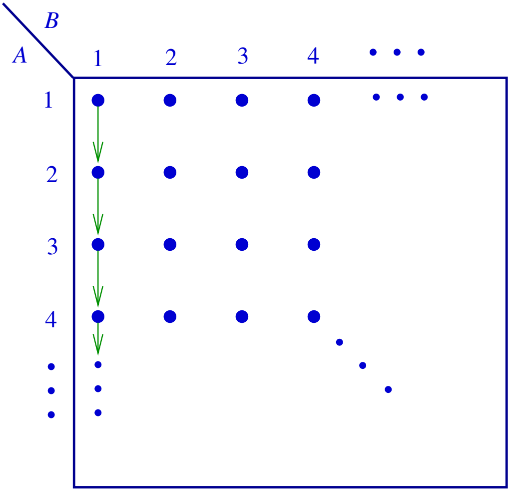

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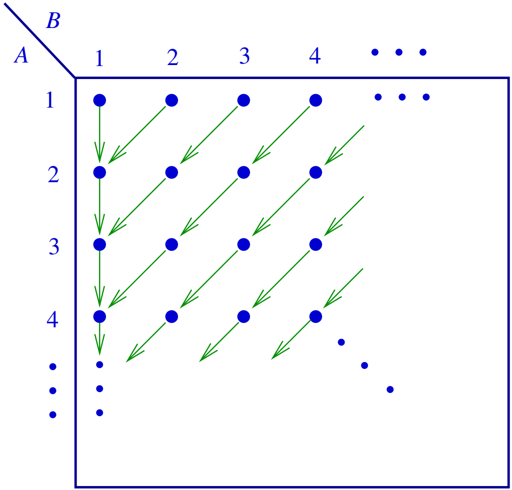

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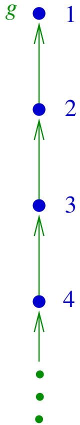

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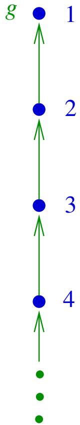

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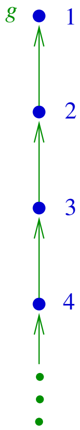

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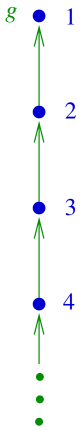

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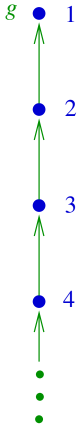

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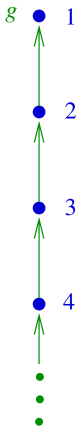

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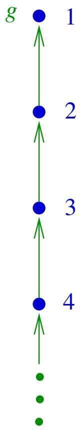

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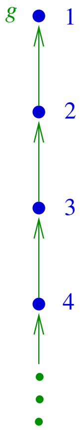

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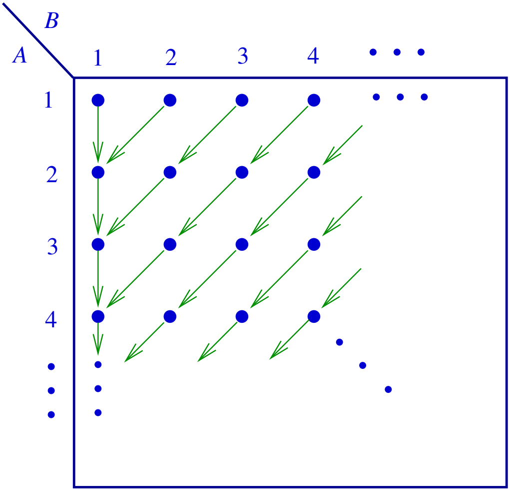

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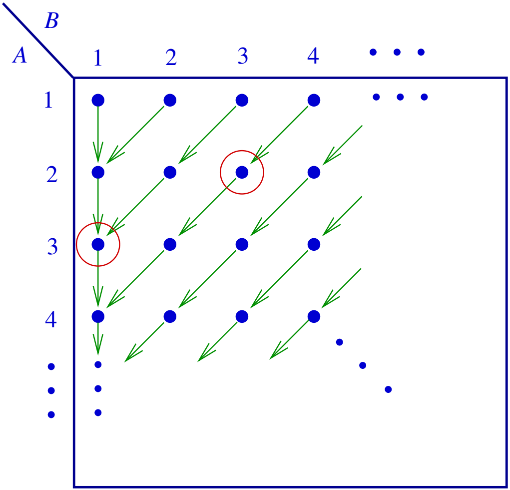

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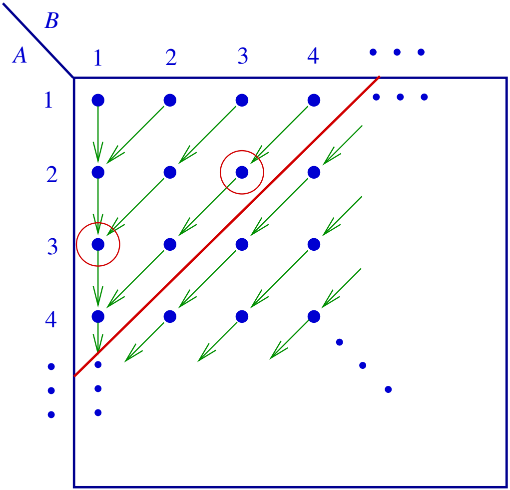

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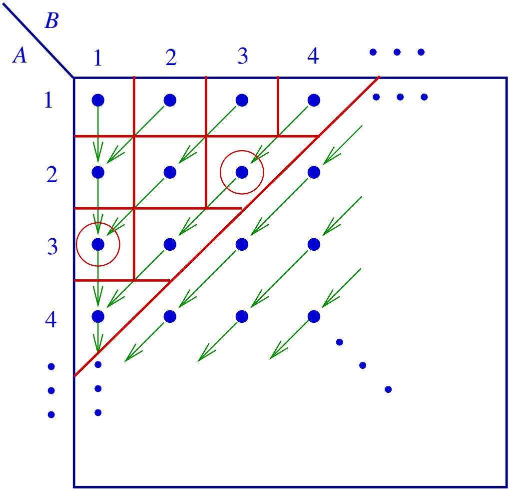

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Other products: wreath product, work in progress with M Quick, M Neunhöffer.