Endomorphism Monoids of Graphs and Partial Orders

Nik Ruskuc
nik@mcs.st-and.ac.uk

School of Mathematics and Statistics, University of St Andrews

Leeds, 3rd August 2007

S.J. Pride:

S.J. Pride:

$$\text{Semigroups} : \text{Groups} = \mathbb{C} : \mathbb{R}$$

S.J. Pride:

$$\text{Semigroups} : \text{Groups} \approx \mathbb{C} : \mathbb{R}$$

S.J. Pride:

$$\text{Semigroups : Groups} \sim \mathbb{C} : \mathbb{R}$$

S.J. Pride:

$$\text{Semigroups} : \text{Groups} \sim \mathbb{C} : \mathbb{R}$$

Groups	:	permutations
Semigroups	:	mappings

S.J. Pride:

$$\text{Semigroups} : \text{Groups} \sim \mathbb{C} : \mathbb{R}$$

Groups	:	automorphisms
Semigroups	:	endomorphisms

Morphisms

Definition

Let $\mathcal{X} = (X;\ R_i\ (i \in I))$ be a relational structure. An endomorphism is a mapping $\theta : X \to X$ which respects all the relations R_i, i.e.

$$(x_1, \dots, x_k) \in R_i \Rightarrow (x_1\theta, \dots, x_k\theta) \in R_i.$$

An automorphism is a bijective endomorphism θ for which θ^{-1} is also an endomorphism.

Remark

For posets and graphs the above becomes:

$$x \leq y \Rightarrow x\theta \leq y\theta,$$
$$x \sim y \Rightarrow x\theta \sim y\theta.$$

Morphisms

$\text{End}(\mathcal{X})$ = the endomorphism monoid of $\mathcal{X}$
$\text{Aut}(\mathcal{X})$ = the automorphism group of $\mathcal{X}$

General Problem

For a given $\mathcal{X}$, how are $\text{End}(\mathcal{X})$ and $\text{Aut}(\mathcal{X})$, and their properties, related?

Trans(X) and Sym(X)

Let $\mathcal{E} = \mathcal{E}(X)$ be a trivial relational structure on X.
Trans(X) = End($\mathcal{E}$) – the full transformation monoid on X
Sym(X) = Aut($\mathcal{E}$) – the symmetric group on X

Facts

- $|\mathrm{Trans}(n)| = n^n$, $\mathrm{Sym}(n) = n!$
- (X infinite) $|\mathrm{Trans}(X)| = |\mathrm{Sym}(X)| = 2^{|X|}$

Finite chains

C_n: $1 < 2 < \ldots < n$

Facts

- $\mathrm{Aut}(C_n) = \{\mathrm{id}\}$
- $|\mathrm{End}(C_n)| = \binom{2n-1}{n-1}$

Rank

Definition

The rank of a semigroup S is the smallest number of elements needed to generate S; notation: $\text{rank}(S)$.

Facts

- $\text{rank}(\text{Sym}(n)) = 2$
- $\text{rank}(\text{Trans}(n)) = 3$
- $\text{rank}(\text{End}(C_n)) = n$

Relative ranks

Definition

Let S be a semigroup, and let T be a subsemigroup of S. The relative rank of S modulo T (denoted $\mathrm{rank}(S : T)$) is the smallest size of a set A such that $S = \langle T \cup A \rangle$.

Example

$\mathrm{rank}(\mathrm{Trans}(n) : \mathrm{Sym}(n)) = 1$

Example: $\mathrm{rank}(\mathrm{Trans}(X) : \mathrm{Sym}(X))$

Proposition (Higgins, Howie, NR 98)

For X infinite, $\mathrm{rank}(\mathrm{Trans}(X) : \mathrm{Sym}(X)) = 2$.

Example: $\text{rank}(\text{Trans}(X) : \text{Sym}(X))$

Proposition (Higgins, Howie, NR 98)

For X infinite, $\text{rank}(\text{Trans}(X) : \text{Sym}(X)) = 2$.

Remark

Semigroups : Groups $= \mathbb{C} : \mathbb{R}$

Example: $\mathrm{rank}(\mathrm{Trans}(X) : \mathrm{Sym}(X))$

Proposition (Higgins, Howie, NR 98)

For X infinite, $\mathrm{rank}(\mathrm{Trans}(X) : \mathrm{Sym}(X)) = 2$.

Remark

$\mathrm{rank}(\mathrm{Trans}(X) : \mathrm{Sym}(X)) = [\mathbb{C} : \mathbb{R}]$:-)

Example: $\mathrm{rank}(\mathrm{Trans}(X) : \mathrm{Sym}(X))$

Proposition (Higgins, Howie, NR 98)

For X infinite, $\mathrm{rank}(\mathrm{Trans}(X) : \mathrm{Sym}(X)) = 2$.

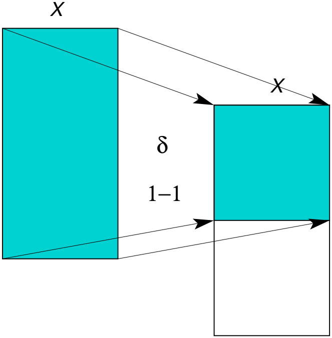
Proof.

Example: $\mathrm{rank}(\mathrm{Trans}(X) : \mathrm{Sym}(X))$

Proposition (Higgins, Howie, NR 98)

For X infinite, $\mathrm{rank}(\mathrm{Trans}(X) : \mathrm{Sym}(X)) = 2$.

Proof. ($\leq$)

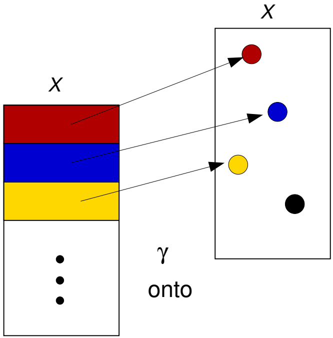


Example: rank(Trans(X) : Sym(X))

Proposition (Higgins, Howie, NR 98)

For X infinite, $\mathrm{rank}(\mathrm{Trans}(X) : \mathrm{Sym}(X)) = 2$.

Proof. ($\leq$)

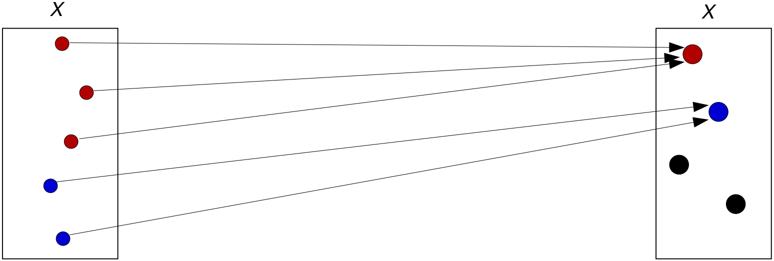


Example: rank(Trans(X) : Sym(X))

Proposition (Higgins, Howie, NR 98)

For X infinite, $\mathrm{rank}(\mathrm{Trans}(X) : \mathrm{Sym}(X)) = 2$*.*

Proof. ($\leq$)

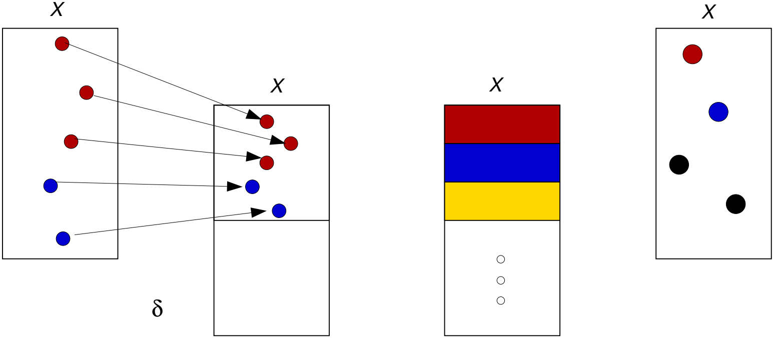


Example: rank(Trans(X) : Sym(X))

Proposition (Higgins, Howie, NR 98)

For X infinite, $\mathrm{rank}(\mathrm{Trans}(X) : \mathrm{Sym}(X)) = 2$*.*

Proof. ($\leq$)

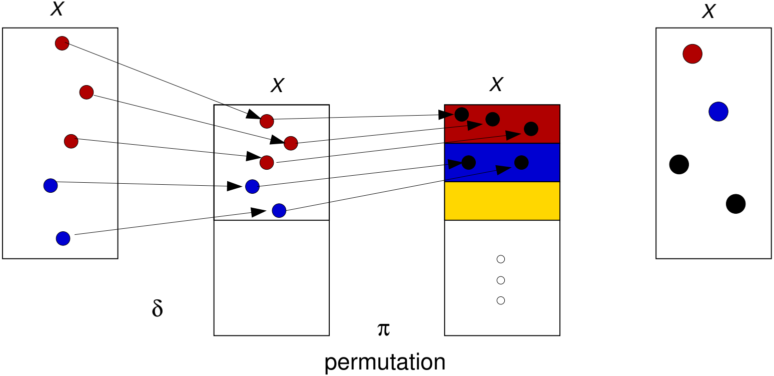


Example: rank(Trans(X) : Sym(X))

Proposition (Higgins, Howie, NR 98)

For X infinite, $\text{rank}(\text{Trans}(X) : \text{Sym}(X)) = 2$*.*

Proof. ($\leq$)

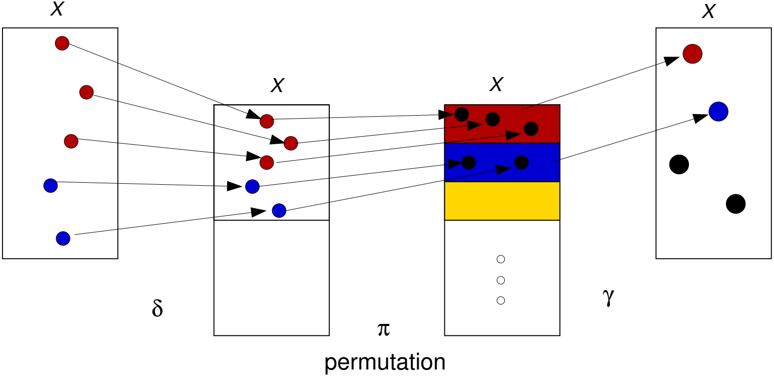


Example: rank(Trans(X) : Sym(X))

Proposition (Higgins, Howie, NR 98)

For X infinite, $\mathrm{rank}(\mathrm{Trans}(X) : \mathrm{Sym}(X)) = 2$.

Proof. ($\leq$)



Example: $\text{rank}(\text{Trans}(X) : \text{Sym}(X))$

Proposition (Higgins, Howie, NR 98)

For X infinite, $\text{rank}(\text{Trans}(X) : \text{Sym}(X)) = 2$.

Proof.

$(\geq)$ Suppose $\text{Trans}(X) = \langle \text{Sym}(X), \tau \rangle$.

- If τ is not injective, then every injection in $\langle \text{Sym}(X), \tau \rangle$ is a bijection.
- If τ is not surjective, then every surjection in $\langle \text{Sym}(X), \tau \rangle$ is a bijection.

Some more relative ranks

- $\text{rank}(\text{Trans}(X) : \langle E(\text{Trans}(X))\rangle) = 2$
- $\text{rank}(\text{Trans}(X) : \text{Inj}(X)) = 1$
- $\text{rank}(\text{Trans}(X) : \text{Surj}(X)) = 1$

Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of $\text{Trans}(X)$ *(X infinite) is contained in a 2-generated subsemigroup of* $\text{Trans}(X)$*.*

Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of $\mathrm{Trans}(X)$ *(X infinite) is contained in a 2-generated subsemigroup of* $\mathrm{Trans}(X)$.

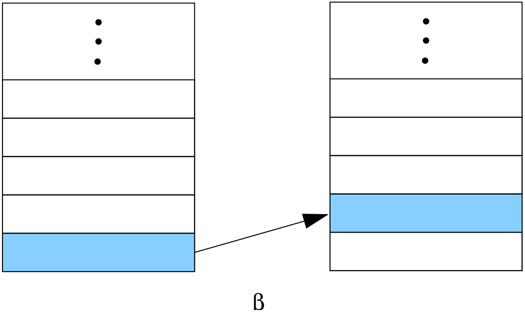
Proof. [Banach 35]

Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X).

Proof. [Banach 35]

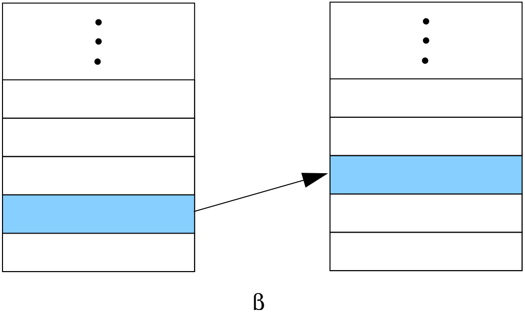


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X).

Proof. [Banach 35]

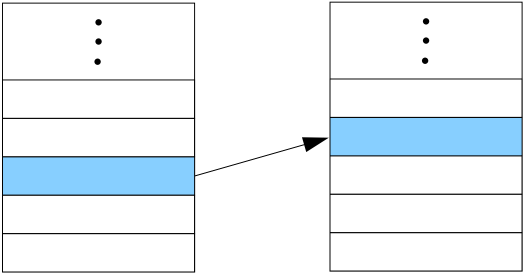


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X).

Proof. [Banach 35]



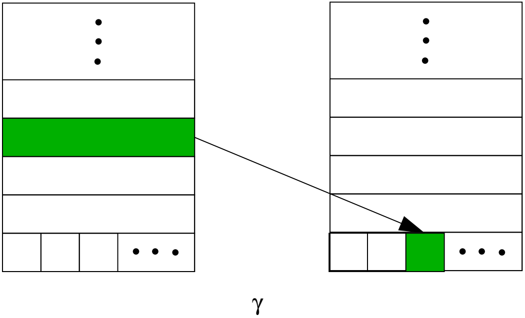
ß

Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X)*.*

Proof. [Banach 35]

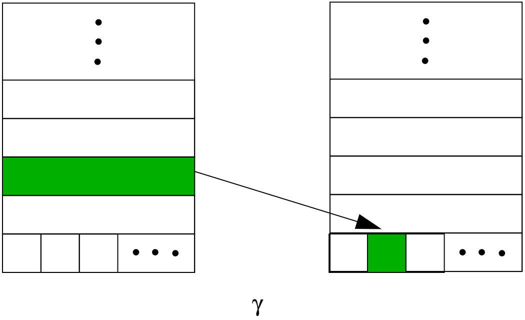


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X).

Proof. [Banach 35]

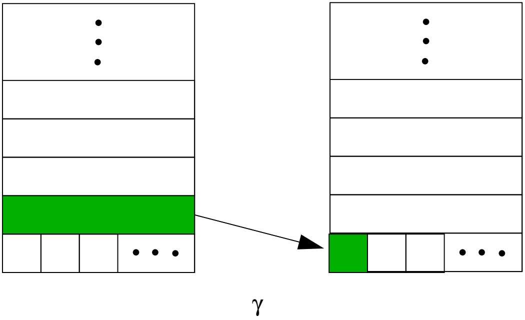


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X).

Proof. [Banach 35]

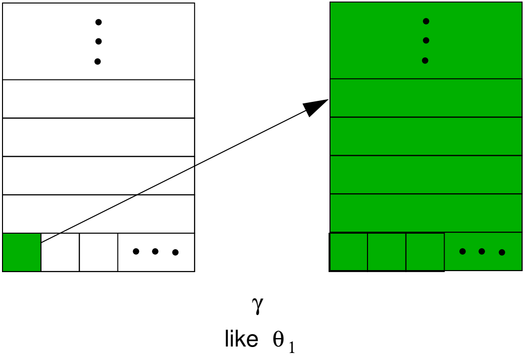


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X)*.*

Proof. [Banach 35]

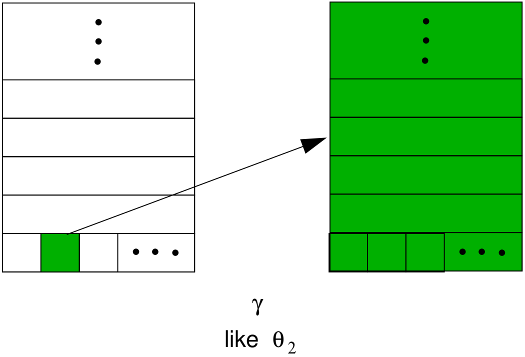


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X).

Proof. [Banach 35]

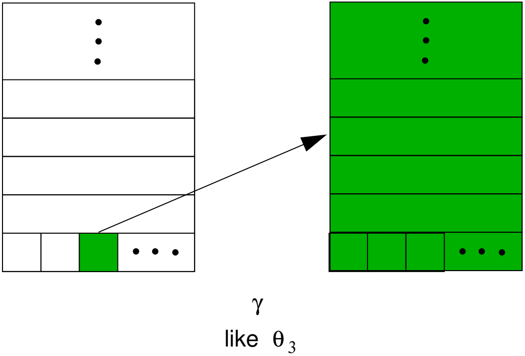


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X).

Proof. [Banach 35]

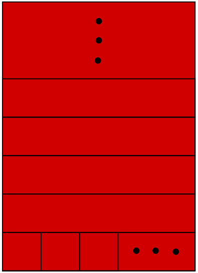


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X)*.*

Proof. [Banach 35]



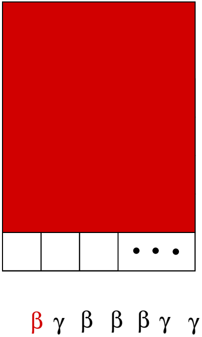
β γ β β β γ γ

Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X).

Proof. [Banach 35]

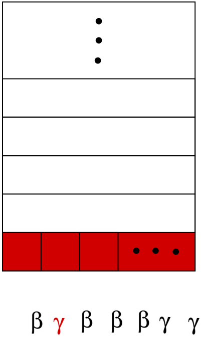


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X).

Proof. [Banach 35]

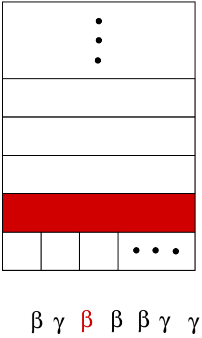


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X).

Proof. [Banach 35]

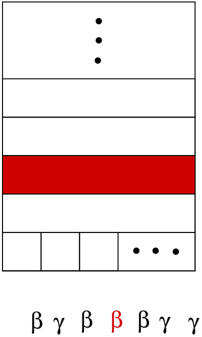


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X)*.*

Proof. [Banach 35]

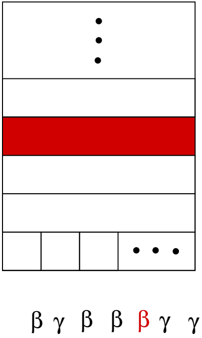


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X).

Proof. [Banach 35]

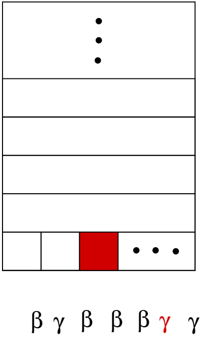


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X).

Proof. [Banach 35]

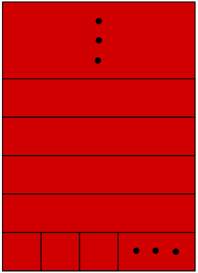


Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X).

Proof. [Banach 35]



$\beta\ \gamma\ \beta\ \beta\ \beta\ \gamma\ \gamma = \theta_3$

Sierpinski's Theorem

Theorem (Sierpinski 35)

Every countable subset of Trans(X) *(X infinite) is contained in a 2-generated subsemigroup of* Trans(X)*.*

Corollary

The relative rank of any subsemigroup of Trans(X) *is* 0, 1, 2 *or uncountable.*

Proof. Suppose rank(Trans(X) : S) $= \aleph_0$, with Trans(X) $= \langle S, \tau_1, \tau_2, \ldots \rangle$. By Sierpinski's Theorem, $\tau_1, \tau_2, \ldots \in \langle \alpha, \beta \rangle$ for some α, β. But then Trans(X) $= \langle S, \alpha, \beta \rangle$.

Sierpinski/Galvin

Theorem (Galvin 93)

Every countable subset of $\mathrm{Sym}(X)$ *(*X *infinite) is contained in a 2-generator subgroup of* $\mathrm{Sym}(X)$*.*

Corollary

The relative rank of any subgroup G *of* $\mathrm{Sym}(X)$ *is* 0, 1 *or uncountable.*

Remark

Where has 2 disappeared to?

Partial bijections

$\mathrm{SymInv}(X)$ – the monoid of all partial bijections of X, i.e. bijections between subsets of X (the symmetric inverse monoid)

Theorem (Mitchell 02)

Every countable subset of $\mathrm{SymInv}(X)$ *(X infinite) is contained in a 2-generated inverse submonoid.*

Corollary

The relative rank of any inverse subsemigroup S of $\mathrm{SymInv}(X)$ *is* 0, 1, 2 *or uncountable.*

Question

Does there exist an inverse subsemigroup of $\mathrm{SymInv}(X)$ with relative rank 2?

Lineraly ordered set $\mathbb{N}$

Theorem (Higgins, Howie, Mitchell, NR 03)

$\text{rank}(\text{Trans}(\mathbb{N}) : \text{End}(\mathbb{N})) = 1$.

Lineraly ordered set $\mathbb{N}$

Theorem (Higgins, Howie, Mitchell, NR 03)

$\text{rank}(\text{Trans}(\mathbb{N}) : \text{End}(\mathbb{N})) = 1$.

Proof. Let $\delta : \mathbb{N} \to \mathbb{N}$ be any onto, infinite-to-one, mapping.

Lineraly ordered set $\mathbb{N}$

Theorem (Higgins, Howie, Mitchell, NR 03)

$\text{rank}(\text{Trans}(\mathbb{N}) : \text{End}(\mathbb{N})) = 1.$

Proof. Let $\delta : \mathbb{N} \to \mathbb{N}$ be any onto, infinite-to-one, mapping.
Let $\gamma \in \text{Trans}(\mathbb{N})$ be arbitrary.

Lineraly ordered set $\mathbb{N}$

Theorem (Higgins, Howie, Mitchell, NR 03)

$\mathrm{rank}(\mathrm{Trans}(\mathbb{N}) : \mathrm{End}(\mathbb{N})) = 1$.

Proof. Let $\delta : \mathbb{N} \to \mathbb{N}$ be any onto, infinite-to-one, mapping.
Let $\gamma \in \mathrm{Trans}(\mathbb{N})$ be arbitrary.
Define $\epsilon \in \mathrm{End}(\mathbb{N})$ inductively by:

Lineraly ordered set $\mathbb{N}$

Theorem (Higgins, Howie, Mitchell, NR 03)

$\text{rank}(\text{Trans}(\mathbb{N}) : \text{End}(\mathbb{N})) = 1$.

Proof. Let $\delta : \mathbb{N} \to \mathbb{N}$ be any onto, infinite-to-one, mapping.
Let $\gamma \in \text{Trans}(\mathbb{N})$ be arbitrary.
Define $\epsilon \in \text{End}(\mathbb{N})$ inductively by:
$k\epsilon \in (k\gamma)\delta^{-1}$ and $k\epsilon > (k-1)\epsilon$.

Lineraly ordered set $\mathbb{N}$

Theorem (Higgins, Howie, Mitchell, NR 03)

$\mathrm{rank}(\mathrm{Trans}(\mathbb{N}) : \mathrm{End}(\mathbb{N})) = 1.$

Proof. Let $\delta : \mathbb{N} \to \mathbb{N}$ be any onto, infinite-to-one, mapping.
Let $\gamma \in \mathrm{Trans}(\mathbb{N})$ be arbitrary.
Define $\epsilon \in \mathrm{End}(\mathbb{N})$ inductively by:
$k\epsilon \in (k\gamma)\delta^{-1}$ and $k\epsilon > (k-1)\epsilon$.
Then $\gamma = \epsilon\delta \in \langle \mathrm{End}(\mathbb{N}), \delta \rangle$.

Linearly ordered sets

Higgins, Mitchell, NR 03

Theorem

If L is a countable linearly ordered set then $\text{rank}(\text{Trans}(L) : \text{End}(L)) = 1$.

Theorem

If L is a well-ordered set then $\text{rank}(\text{Trans}(L) : \text{End}(L)) = 1$.

Example

$\text{rank}(\text{Trans}(\mathbb{R}) : \text{End}(\mathbb{R}))$ is uncountable.

Question

Does there exist an infinite linearly ordered set L such that $\text{rank}(\text{Trans}(L) : \text{End}(L)) = 2$?

Question

Classify, in order-theoretic terms, all linearly ordered sets L with finite relative ranks in $\text{Trans}(L)$.

Partially ordered sets

Definition

Let P be a partially ordered set. For $x \in P$ write:

$$x^{\wedge} = \{p \in P \,:\, x \leq p\},\ x^{\vee} = \{p \in P \,:\, p \geq x\}$$

Partially ordered sets

Definition

Let P be a partially ordered set. For $x \in P$ write:

$$x^{\wedge} = \{p \in P \,:\, x \le p\},\ x^{\vee} = \{p \in P \,:\, p \ge x\}$$

Theorem (Higgins, Mitchell, Morayne, NR 06)

Let P be a countably infinite poset, and let c be the number of connected components of P.

Partially ordered sets

Definition

Let P be a partially ordered set. For $x \in P$ write:

$$x^{\wedge} = \{p \in P \,:\, x \leq p\},\ x^{\vee} = \{p \in P \,:\, p \geq x\}$$

Theorem (Higgins, Mitchell, Morayne, NR 06)

Let P be a countably infinite poset, and let c be the number of connected components of P.
(I) *Suppose c is finite. Then* $\operatorname{rank}(\operatorname{Trans}(P), \operatorname{End}(P)) \leq 2$ *if and only if there exists $x \in P$ such that at least one of $x^{\wedge}$ or $x^{\vee}$ is infinite.*

Partially ordered sets

Definition

Let P be a partially ordered set. For $x \in P$ write:

$$x^{\wedge} = \{p \in P \,:\, x \le p\},\; x^{\vee} = \{p \in P \,:\, p \ge x\}$$

Theorem (Higgins, Mitchell, Morayne, NR 06)

Let P be a countably infinite poset, and let c be the number of connected components of P.

(I) *Suppose c is finite. Then* $\operatorname{rank}(\operatorname{Trans}(P), \operatorname{End}(P)) \le 2$ *if and only if there exists $x \in P$ such that at least one of $x^{\wedge}$ or $x^{\vee}$ is infinite.*

(II) *If c is infinite and P is not an antichain, then* $\operatorname{rank}(\operatorname{Trans}(P), \operatorname{End}(P)) = 1$.

Partially ordered sets

Definition

Let P be a partially ordered set. For $x \in P$ write:

$$x^{\wedge} = \{p \in P : x \leq p\}, \; x^{\vee} = \{p \in P : p \geq x\}$$

Theorem (Higgins, Mitchell, Morayne, NR 06)

Let P be a countably infinite poset, and let c be the number of connected components of P.
(I) *Suppose c is finite. Then* $\mathrm{rank}(\mathrm{Trans}(P), \mathrm{End}(P)) \leq 2$ *if and only if there exists $x \in P$ such that at least one of $x^{\wedge}$ or $x^{\vee}$ is infinite.*
(II) *If c is infinite and P is not an antichain, then* $\mathrm{rank}(\mathrm{Trans}(P), \mathrm{End}(P)) = 1$.
(III) *If P is an antichain, then* $\mathrm{rank}(\mathrm{Trans}(P), \mathrm{End}(P)) = 0$:-)

Partially ordered sets

Corollary

$\mathrm{rank}(\mathrm{Trans}(P), \mathrm{End}(P)) \leq 2$ *provided any of the following hold:*

- P *has a smallest element;*
- P *has a largest element;*
- P *is a lattice.*

A poset with relative rank 2 (I)

Higgins, Mitchell, Morayne, NR 06

Lemma

If P is a poset with no non-trivial mono- or epi-morphisms then $\text{rank}(\text{Trans}(P), \text{End}(P)) \geq 2$.

A poset with relative rank 2 (I)

Higgins, Mitchell, Morayne, NR 06

Lemma

If P is a poset with no non-trivial mono- or epi-morphisms then $\text{rank}(\text{Trans}(P), \text{End}(P)) \geq 2$.

Proof. Suppose $\text{Trans}(P) = \langle \text{End}(P), \mu \rangle$.

A poset with relative rank 2 (I)

Higgins, Mitchell, Morayne, NR 06

Lemma

If P is a poset with no non-trivial mono- or epi-morphisms then $\mathrm{rank}(\mathrm{Trans}(P), \mathrm{End}(P)) \geq 2$.

Proof. Suppose $\mathrm{Trans}(P) = \langle \mathrm{End}(P), \mu \rangle$.
Let $\pi \in \mathrm{Sym}(P)$; write $\pi = \gamma_1 \gamma_2 \ldots \gamma_n$, $\gamma_i \in \mathrm{End}(P) \cup \{\mu\}$.

A poset with relative rank 2 (I)

Higgins, Mitchell, Morayne, NR 06

Lemma

If P is a poset with no non-trivial mono- or epi-morphisms then $\mathrm{rank}(\mathrm{Trans}(P), \mathrm{End}(P)) \geq 2$.

Proof. Suppose $\mathrm{Trans}(P) = \langle \mathrm{End}(P), \mu \rangle$.
Let $\pi \in \mathrm{Sym}(P)$; write $\pi = \gamma_1\gamma_2 \ldots \gamma_n$, $\gamma_i \in \mathrm{End}(P) \cup \{\mu\}$.
γ_1 is injective; γ_n is surjective.

A poset with relative rank 2 (I)

Higgins, Mitchell, Morayne, NR 06

Lemma

If P is a poset with no non-trivial mono- or epi-morphisms then $\mathrm{rank}(\mathrm{Trans}(P), \mathrm{End}(P)) \geq 2$.

Proof. Suppose $\mathrm{Trans}(P) = \langle \mathrm{End}(P), \mu \rangle$.
Let $\pi \in \mathrm{Sym}(P)$; write $\pi = \gamma_1 \gamma_2 \ldots \gamma_n$, $\gamma_i \in \mathrm{End}(P) \cup \{\mu\}$.
γ_1 is injective; γ_n is surjective.
Hence $\gamma_1 = \gamma_n = \mu \in \mathrm{Sym}(P)$.

A poset with relative rank 2 (I)

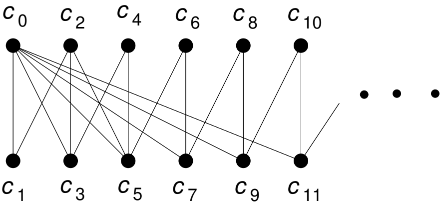
Higgins, Mitchell, Morayne, NR 06

Lemma

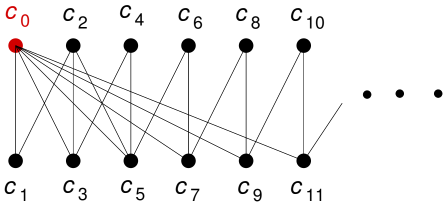
If P is a poset with no non-trivial mono- or epi-morphisms then $\mathrm{rank}(\mathrm{Trans}(P), \mathrm{End}(P)) \geq 2$.

Proof. Suppose $\mathrm{Trans}(P) = \langle \mathrm{End}(P), \mu \rangle$.
Let $\pi \in \mathrm{Sym}(P)$; write $\pi = \gamma_1 \gamma_2 \ldots \gamma_n$, $\gamma_i \in \mathrm{End}(P) \cup \{\mu\}$.
γ_1 is injective; γ_n is surjective.
Hence $\gamma_1 = \gamma_n = \mu \in \mathrm{Sym}(P)$.
By induction $\pi = \mu^n$, a contradiction.

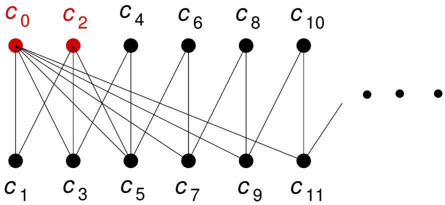
Poset with no monomorphisms



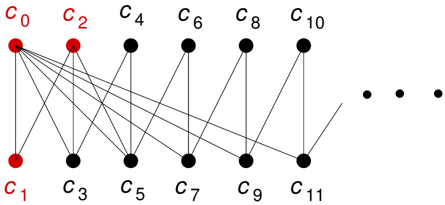
Poset with no monomorphisms



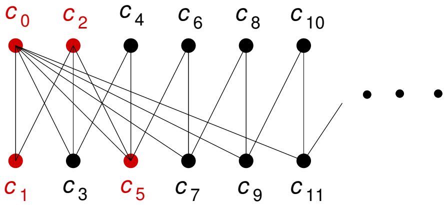
Poset with no monomorphisms



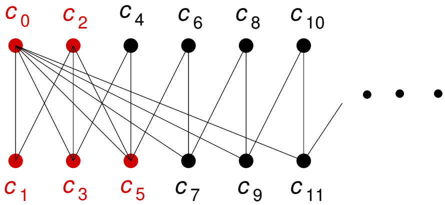
Poset with no monomorphisms



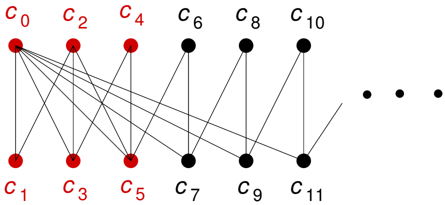
Poset with no monomorphisms



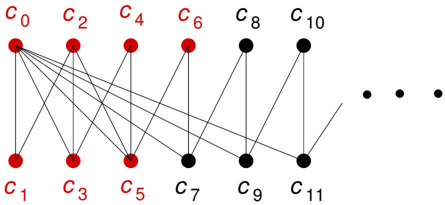
Poset with no monomorphisms



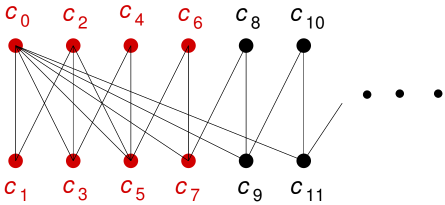
Poset with no monomorphisms



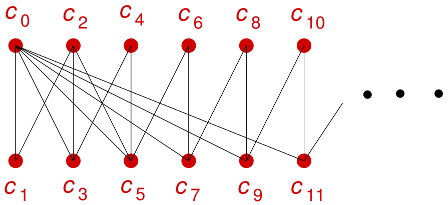
Poset with no monomorphisms



Poset with no monomorphisms



Poset with no monomorphisms



Poset with no epimorphisms

Poset with no epimorphisms

Let $A = \{a_1, a_2, \ldots\}$ be a countably infinite set.

Poset with no epimorphisms

Let $A = \{a_1, a_2, \ldots\}$ be a countably infinite set.
Let $\mathcal{E}$ be the set of finite subsets E of A satisfying:
$a_n \in E \Rightarrow 2 \leq |E| \leq n + 1$.

Poset with no epimorphisms

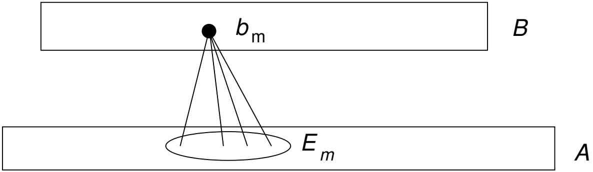
Let $A = \{a_1, a_2, \ldots\}$ be a countably infinite set.
Let $\mathcal{E}$ be the set of finite subsets E of A satisfying:
$a_n \in E \Rightarrow 2 \leq |E| \leq n+1$.
Enumerate $\mathcal{E}$: E_1, E_2, ...

Poset with no epimorphisms

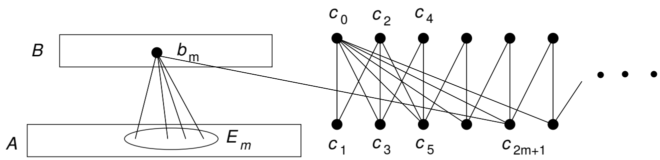
Let $A = \{a_1, a_2, \ldots\}$ be a countably infinite set.
Let $\mathcal{E}$ be the set of finite subsets E of A satisfying:
$a_n \in E \Rightarrow 2 \leq |E| \leq n + 1$.
Enumerate $\mathcal{E}$: E_1, E_2, $\ldots$
Let $B = \{b_1, b_2, \ldots\}$.

Poset with no epimorphisms

Let $A = \{a_1, a_2, \ldots\}$ be a countably infinite set.
Let $\mathcal{E}$ be the set of finite subsets E of A satisfying:
$a_n \in E \Rightarrow 2 \leq |E| \leq n + 1$.
Enumerate $\mathcal{E}$: E_1, E_2, ...
Let $B = \{b_1, b_2, \ldots\}$.
A partial order on $A \cup B$: $b_m > a_n \Leftrightarrow a_n \in E_m$.



A poset with relative rank 2 (II)



Proposition (Higgins, Mitchell, Morayne, NR 06)

The above poset P satisfies the conditions from the part (I) of the Theorem, but has no non-trivial mono- or epi-morphisms; consequently $\mathrm{rank}(\mathrm{Trans}(P) : \mathrm{End}(P)) = 2$.

Random graph

R – the random graph

Theorem (Truss 85)

Every countable group embeds into $\mathrm{Aut}(R)$.

Random graph

R – the random graph

Theorem (Truss 85)

Every countable group embeds into $\mathrm{Aut}(R)$.

Theorem (Bonato, Delic, Dolinka , to appear)

Every countable monoid embeds into $\mathrm{End}(R)$.

Random graph

Theorem (Truss 85)

$\mathrm{Aut}(R)$ *is simple.*

Random graph

Theorem (Truss 85)

Aut(R) *is simple.*

Theorem (Delic, Dolinka 04)

End(R) *has uncountably many ideals.*

Remark

To every ideal I of a monoid M there corresponds the Rees congruence $\Phi_I \cup \Delta_M$ on M; but, not every congruence is a Rees congruence.

Random graph

Theorem (Truss 85)

Aut(R) *is simple.*

Theorem (Delic, Dolinka 04)

End(R) *has uncountably many ideals.*

Remark

To every ideal I of a monoid M there corresponds the Rees congruence $\Phi_I \cup \Delta_M$ on M; but, not every congruence is a Rees congruence.

Question

How many non-Rees congruences does End(R) have?

Random graph

Theorem (Truss 85)

Aut(R) *is simple.*

Theorem (Delic, Dolinka 04)

End(R) *has uncountably many ideals.*

Remark

To every ideal I of a monoid M there corresponds the Rees congruence $\Phi_I \cup \Delta_M$ on M; but, not every congruence is a Rees congruence.

Question

How many non-Rees congruences does End(R) have?

Question

How many congruences does Trans($\mathbb{N}$) have?

Random graph

Definition

The semigroup S is said to have Sierpinski index n if every countable subset of S is contained in an n-generated subsemigroup of S.

Random graph

Definition

The semigroup S is said to have Sierpinski index n if every countable subset of S is contained in an n-generated subsemigroup of S.

Question

Does $\mathrm{Aut}(R)$ have a finite Sierpinski index?

Random graph

Definition

The semigroup S is said to have Sierpinski index n if every countable subset of S is contained in an n-generated subsemigroup of S.

Question

Does $\mathrm{Aut}(R)$ have a finite Sierpinski index?

Question

Does $\mathrm{End}(R)$ have a finite Sierpinski index?

Random graph

Definition

The semigroup S is said to have Sierpinski index n if every countable subset of S is contained in an n-generated subsemigroup of S.

Question

Does $\mathrm{Aut}(R)$ have a finite Sierpinski index?

Question

Does $\mathrm{End}(R)$ have a finite Sierpinski index?

Facts

$\mathrm{rank}(\mathrm{Sym}(R) : \mathrm{Aut}(R)) = 1$, $\mathrm{rank}(\mathrm{Trans}(R) : \mathrm{Aut}(R)) = 2$, $\mathrm{rank}(\mathrm{Trans}(R) : \mathrm{End}(R)) = 1$.

Question

$\mathrm{rank}(\mathrm{End}(R) : \mathrm{Aut}(R)) =?$

Homogeneous structures

Question

Let H be a homogeneous relational structure. Find general/interesting conditions under which:

- rank(Sym(H) : Aut(H)) is finite;
- rank(Trans(H) : End(H)) is finite;
- rank(End(H) : Aut(H)) is finite;
- End(H) and/or Aut(H) have finite Sierpinski index.

Homomorphism homogeneous structures

P.J. Cameron, J. Nesetril, Homomorphism-homogeneous relational structures, preprint
P.J. Cameron, D. Lockett, Posets, homomorphisms and homogeneity, preprint

Question

Investigate algebraic and combinatorial properties of endomorphism monoids of these new types of homogeneous structures, and how they relate to the corresponding automorphism groups.