Growth of Generating Sets of Direct Powers

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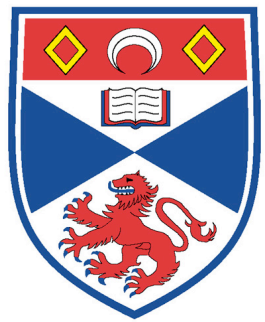

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The **d**-sequence and Types of Growth

Let A be an algebra(ic structure).

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Finite (Semi)groups

J. Wiegold (1974–87)

S (a finite semigroup)	$\mathbf{d}(S)$
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non-group monoid	linear
non-monoid semigroup	exponential

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Theorem

Let A be a finite simple algebra in a congruence permutable variety.

(1) *If A is functionally complete then $\mathbf{d}(A)$ grows logarithmically.*

(2) *If A is polynomially equivalent to a simple module then $\mathbf{d}(A)$ grows linearly.*

Growth for Finite Classical Structures

Theorem

If A is a finite group, ring, module, algebra or Lie algebra, then $\mathbf{d}(A)$ grows either logarithmically or linearly.

Remark

Those with logarithmic growth are:

- perfect groups,
- rings with 1,
- algebras with 1,
- perfect Lie algebras.

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Wiegold & Wilson (1978), Stewart & Wiegold (1989).

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Theorem (Robertson, NR, Wiegold)

Let S be an infinite semigroup. If $S \times S$ is finitely generated, then S^n is finitely generated for all n, and $\mathbf{d}(S)$ grows at worst exponentially.

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Theorem (St Andrews Summer School)

For the polycyclic monoid

$$P_k = \langle b_i, c_i \ (i = 1, \ldots, k) \mid b_i c_i = 1, \ b_i c_j = 0 \ (i \neq j) \rangle$$

we have

$$d(P_k^n) = nk + k - 1.$$

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Theorem

There exists an infinite finitely generated semigroup S (without identity) such that $\mathbf{d}(S)$ is constant.

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- A group G for which the growth of $\mathbf{d}(G)$ is strictly between constant and logarithmic?
- A semigroup S for which the growth of $\mathbf{d}(S)$ is strictly between linear and exponential?
- A finite algebra A in a congruence permutable variety for which the growth of $\mathbf{d}(A)$ is strictly between logarithmic and linear?

Future Directions

- Other structures: lattices, tournaments, Steiner triple systems (A. Geddes, M. Quick, NR).

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- Iterating other constructions: wreath products (M. Neunhöffer, M. Quick, NR).