Growth of Generating Sets of Direct Powers

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The **d**-sequence and Types of Growth

Let A be an algebra(ic structure).

- $d(A)$ = minimal size of a generating set of A.
- $\mathbf{d}(A) = (d(A), d(A \times A), d(A \times A \times A), \ldots)$.

Very, very good	Constant
Good	Logarithmic
Middling	Linear
Poor	Exponential
Positively bad	∞

Examples

Examples

- Cyclic groups: $\mathbf{d}(C_n) = (1, 2, 3, 4, \ldots)$ (middling).
- Alternating groups: $\mathbf{d}(A_5) = (\underbrace{2, \ldots, 2}_{19}, \underbrace{3, \ldots, 3}_{1649}, 4, \ldots)$

  (Hall 1936, good).
- Left zeros: $\mathbf{d}(LZ_2) = (2, 4, 8, 16, \ldots)$ (poor).
- Positive integers: $\mathbf{d}(\mathbb{N}) = (1, \infty, \infty, \ldots)$ (positively bad ☺)

Finite (Semi)groups

J. Wiegold (1974–87)

S (a finite semigroup)	$\mathbf{d}(S)$
perfect group	logarithmic
non-perfect group	linear
non-group monoid	linear
non-monoid semigroup	exponential

A Universal Algebra Excursion

Theorem (Werner; Herrmann)

A finite simple algebra in a congruence permutable variety is either functionally complete or else it is polynomially equivalent to a simple module over a finite ring with 1.

Theorem

Let A be a finite simple algebra in a congruence permutable variety.

(1) *If A is functionally complete then $\mathbf{d}(A)$ grows logarithmically.*

(2) *If A is polynomially equivalent to a simple module then $\mathbf{d}(A)$ grows linearly.*

Growth for Finite Classical Structures

Theorem

If A is a finite group, ring, module, algebra or Lie algebra, then $\mathbf{d}(A)$ grows either logarithmically or linearly.

Remark

Those with logarithmic growth are:

- perfect groups,
- rings with 1,
- algebras with 1,
- perfect Lie algebras.

Wiegold on Infinite Groups

Wiegold & Wilson (1978), Stewart & Wiegold (1989).

G (an infinite group)	$\mathbf{d}(G)$
simple	constant 
non-simple, perfect	constant $\leq \mathbf{d}(G) \leq$ logarithmic 
non-perfect	linear

Infinite Semigroups

Theorem (Robertson, NR, Wiegold)

Let S be an infinite semigroup. If $S \times S$ is finitely generated, then S^n is finitely generated for all n, and $\mathbf{d}(S)$ grows at worst exponentially.

Theorem (St Andrews Summer School)

For the polycyclic monoid

$$P_k = \langle b_i, c_i \ (i = 1, \ldots, k) \mid b_i c_i = 1, \ b_i c_j = 0 \ (i \neq j) \rangle$$

we have

$$d(P_k^n) = nk + k - 1.$$

Cyclic Diagonal Acts

Let S be an infinite semigroup such that

- S is finitely generated;
- There exist $a, b \in S$ such that $S \times S = \{(as, bs) : s \in S\}$.

Fact

Such S exists, and can be chosen: to be finitely presented, to be simple, to have a soluble word problem, to have no identity element, etc.

Fact (Robertson, NR, Thomson)

For every n there exist $a_1, \ldots, a_n \subseteq S$ such that

$$S^n = \{(a_1 s, \ldots, a_n s) : s \in S\}.$$

Theorem

There exists an infinite finitely generated semigroup S (without identity) such that $\mathbf{d}(S)$ is constant.

Some Problems

As of today:

- Compute $\mathbf{d}(M_{k,1})$ for the Thompson–Higman monoids introduced in J-C Birget's talk.

Does there exist:

- An algebra A for which the growth of $\mathbf{d}(A)$ is neither constant, logarithmic, linear, nor exponential?
- A group G for which the growth of $\mathbf{d}(G)$ is strictly between constant and logarithmic?
- A semigroup S for which the growth of $\mathbf{d}(S)$ is strictly between linear and exponential?
- A finite algebra A in a congruence permutable variety for which the growth of $\mathbf{d}(A)$ is strictly between logarithmic and linear?

Future Directions

- Other structures: lattices, tournaments, Steiner triple systems (A. Geddes, M. Quick, NR).
- Iterating other constructions: wreath products (M. Neunhöffer, M. Quick, NR).