$1+1=2$: applications to direct products of semigroups

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Preview: $1+1=2$

> *. . . for I cannot satisfy myself that, when one is added to one, the one to which the addition is made becomes two, or that the two units added together make two by reason of the addition. I cannot understand how, when separated from the other, each of them was one and not two, and now, when they are brought together, the mere juxtaposition or meeting of them should be the cause of their becoming two: neither can I understand how the division of one is the way to make two; for then a different cause would produce the same effect,–as in the former instance the addition and juxtaposition of one to one was the cause of two, in this the separation and subtraction of one from the other would be the cause. Nor am I any longer satisfied that I understand the reason why one or anything else is either generated or destroyed or is at all, but I have in my mind some confused notion of a new method . . .* (Socrates in Plato's Phaedo)

The generic problem

$\mathcal{P}$ – an algebraic property (finiteness condition).

Generic Problem

Find a necessary and sufficient condition (in terms of A and B) for the direct product $A \times B$ to satisfy property $\mathcal{P}$.

Generic Theorem

$A \times B$ satisfies $\mathcal{P}$ iff A and B satisfy $\mathcal{P}$.

Definition

NICE, BORING THEOREM.

Examples

Finiteness, periodicity (for semigroups), finite generation (for groups and monoids).

A, B, $A \times B$

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$$\begin{tikzcd}
A & A \text{ x } B \arrow[l, two heads, "\pi_A : (a,b) \longrightarrow a"'] \arrow[r, two heads, "\pi_B : (a,b) \longrightarrow b"] & B
\end{tikzcd}$$

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Provided e is an idempotent

$\mathbb{N} \times \mathbb{N}$

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Consider the additive semigroup $\mathbb{N} \times \mathbb{N}$ ($0 \notin \mathbb{N}$).

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We say that $(1, n)$ is indecomposable.

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We say that $(1, n)$ is indecomposable.
So, $\{(1, n) : n \in \mathbb{N}\}$ is contained in every generating set.

Finite Generation: Semigroups

Theorem (EF Robertson, NR, J. Wiegold)

Let S, T be infinite semigroups.

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Theorem

Let S, T be semigroups, with S infinite, T finite. $S \times T$ is finitely generated if and only if

(i) *S is finitely generated; and*

(ii) *T has no indecomposable elements.*

Finite Presentability for Semigroups

Question

When is the direct product $S \times T$ of two semigroups finitely presented?

Remarks

- NBT: groups, monoids.
- S, T infinite $\Rightarrow$ no indecomposable elements.

Critical Pairs and Stability

S – a semigroup; $\langle A|R\rangle$ a finite presentation for it.

Fact

Two words u, v over A are equal in S if and only if there is a sequences of applications of relations from R (a deduction) which transforms u into v.

Definition

A pair (u, v) of words is critical if every deduction from u to v contains a word of length smaller than $\min(|u|, |v|)$.

Definition

S is said to be stable if it has no critical pairs.

Remarks

- Definition independent of A, but dependent on R.
- Not constructive.

Stability and Finite Presentability

Theorem (EF Robertson, NR, J Wiegold)

Let S, T be two infinite semigroups. $S \times T$ is finitely presented if and only if

(i) *S and T are (finitely presented) and stable; and*

(ii) *neither S nor T contain indecomposable elements.*

Theorem (EF Robertson, NR, J Wiegold)

Let S be an infinite semigroup, and let T be a finite semigroup. $S \times T$ is finitely presented if and only if

(i) *S is finitely presented; and*

(ii) *T is stable and contains no indecomposable elements.*

Finite Presentability: ‘Good Classes’

Corollary

Suppose that S and T belong to any of the following classes: monoids (including groups), regular semigroups (including inverse semigroups), surjective commutative semigroups $(\dots)$ Then $S \times T$ is finitely presented if and only if S and T are finitely presented.

Some Non-Finitely-Presented Examples

Theorem (I Araujo, NR)

There is an (effective) algorithm which decides whether a finite semigroup is stable.

Example

The four element semigroup

S	a	b	c	0
a	a	a	c	0
b	b	b	c	0
c	0	0	0	0
0	0	0	0	0

is a non-stable semigroup of minimal size. Hence, for example, $S \times \mathbb{Z}$ is finitely generated but not finitely presented.

Wreath products: groups and monoids

Theorem

The wreath product $G \wr H$ of two groups is finitely generated iff G and H are finitely generated.

Theorem (EF Robertson, NR, MR Thomson)

Let A, B be non-trivial monoids, and let U be the group of units of B. The wreath product $A \wr B$ is finitely generated iff both A, B are finitely generated and $B = FU$ for some finite subset $F \subseteq B$.

Wreath products: semigroups

Theorem (EF Robertson, NR, MR Thomson)

Let S, T be non-trivial semigroups, T finite. The wreath product $S \wr T$ is finitely generated iff the following conditions are satisfied:

(i) $SS = S$, $TT = T$;

(ii) *S is finitely generated;*

(iii) *Either S has a finitely generated right diagonal act or T is the union of principal right ideals of its right identities.*

Residual Finiteness: Definition

Definition

An algebraic structure A is residually finite if for any two $a, b \in A$ ($a \neq b$) there is a homomorphism $f : A \to B$, B finite, such that $f(a) \neq f(b)$.

Equivalently, A is residually finite if for any two $a, b \in A$ ($a \neq b$) there exists a congruence ρ with finitely many classes such that $(a, b) \notin \rho$.

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Map: $A \times B \xrightarrow{\pi_A} A \xrightarrow{f} C$, C finite, $f(a) \neq f(c)$.

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Residual finiteness is a hereditary property: If A is residually finite and $B \leq A$ the A is residually finite as well.

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Fact

Residual finiteness is a hereditary property: If A is residually finite and $B \leq A$ the A is residually finite as well.

Proposition

Suppose both A and B contain idempotents. Then $A \times B$ is residually finite iff A and B are residually finite.

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R. Gray, NR

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Let ρ be a congruence on $\mathbb{N} \times S$ with finitely many classes that separates $(2, a)$ and $(2, b)$.

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The equivalence relation λ with classes L_1, L_2, $L_3 \cup L_4 \cup \ldots$ is a congruence with finitely many classes.

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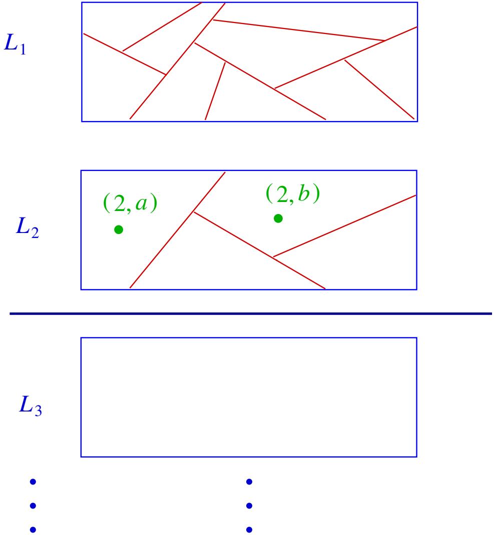
The equivalence relation λ with classes L_1, L_2, $L_3 \cup L_4 \cup \ldots$ is a congruence with finitely many classes.
Intersect ρ and λ to obtain a congruence $\sigma = \rho \cap \lambda$ which has finitely many classes, respects levels 1 and 2, and separates $(2, a)$ and $(2, b)$.

Residual Finiteness: Levels of $\mathbb{N} \times S$



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Claim: $\gamma \subseteq \tau \subseteq \delta$.

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Proof: τ is obtained by taking pairs (xu, yu), $(x, y) \in \gamma$, $u \in S^1$, and closing transitively.

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$$(x, y) \in \gamma \quad \Rightarrow \quad ((1, x), (1, y)) \in \sigma$$

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Hence: τ has finitely many classes,

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A technicality to pass to a congruence.

Residual Finiteness: A Nice, (Not Boring?) Theorem

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Theorem (Gray, NR)

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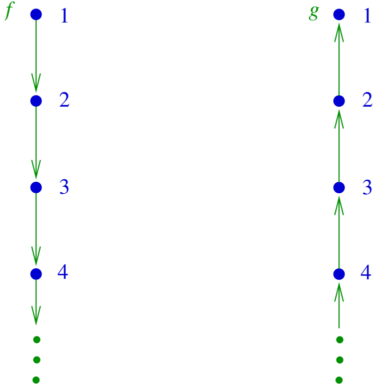
Proof

A semigroup either contains an idempotent or a copy of $\mathbb{N}$.

Residual Finiteness: Unary Algebras

Example

Consider two unary operations on $\mathbb{N}$:



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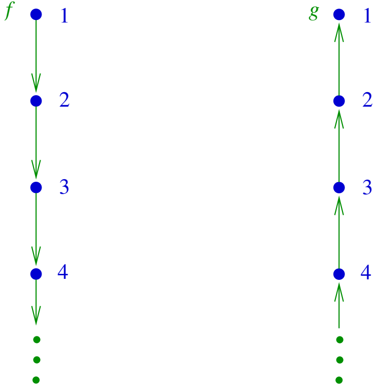

Let $A = (\mathbb{N}, f)$, $B = (\mathbb{N}, g)$.

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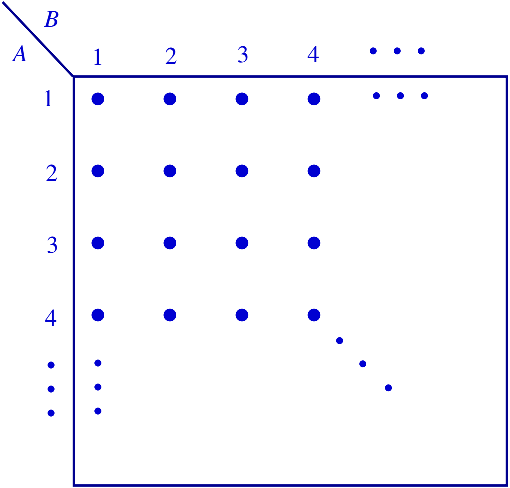

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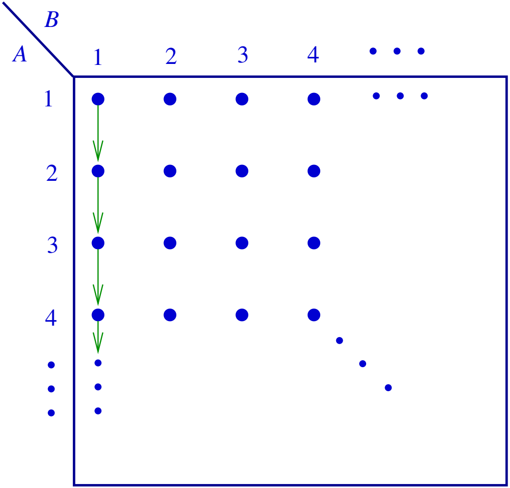

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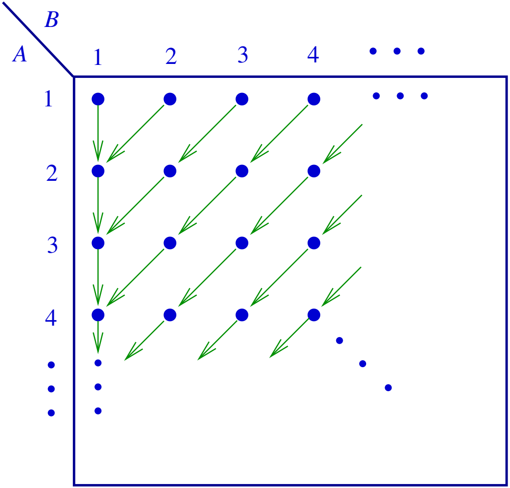

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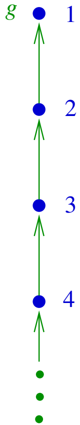

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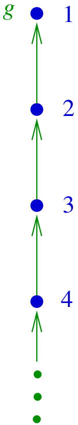

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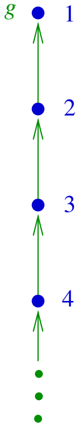

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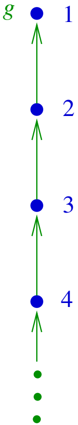

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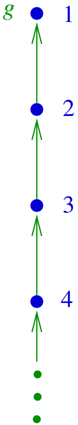

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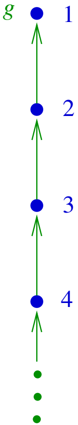

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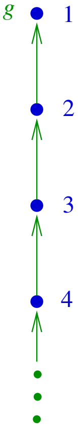
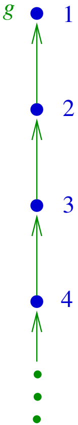
$$\begin{aligned}(m, n) \in \rho \quad &\Rightarrow \quad (m-1, n-1) = (g(m), g(n)) \in \rho \\ &\Rightarrow \quad \ldots \Rightarrow (2, 1) \in \rho,\end{aligned}$$

a contradiction.



Residual Finiteness: Unary Algebras



Lemma

B is not residually finite.

Proof

Suppose ρ is a congruence with finitely many classes separating 1 and 2.
Let $(m, n) \in \rho$, $m > n$.
We have

$$\begin{aligned} (m,n)\in\rho \quad &\Rightarrow \quad (m-1,n-1)=(g(m),g(n))\in\rho \\ &\Rightarrow \quad \ldots \Rightarrow (2,1)\in\rho, \end{aligned}$$

a contradiction.

Lemma

A is residually finite.

Residual Finiteness: Unary Algebras

Lemma

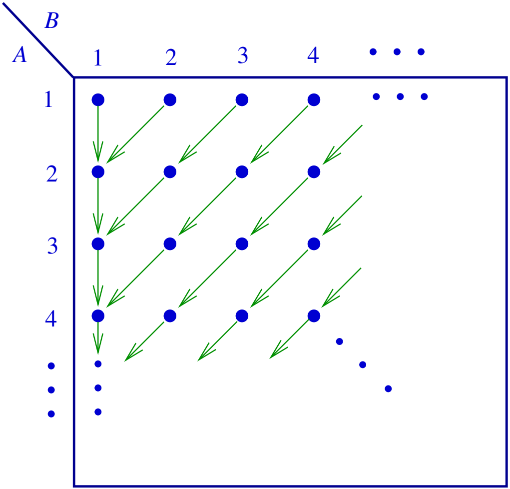
$A \times B$ is residually finite.

Residual Finiteness: Unary Algebras

Lemma

$A \times B$ is residually finite.

Proof

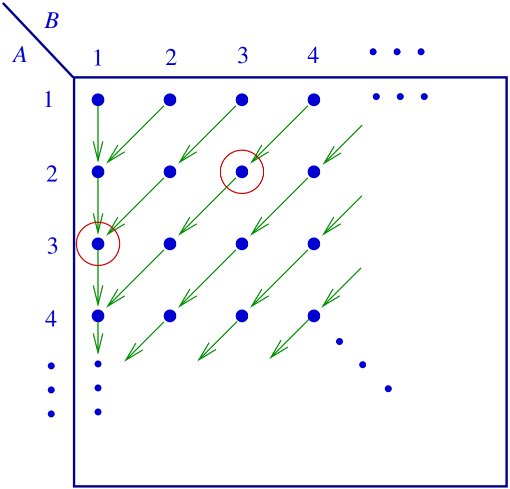


Residual Finiteness: Unary Algebras

Lemma

$A \times B$ is residually finite.

Proof



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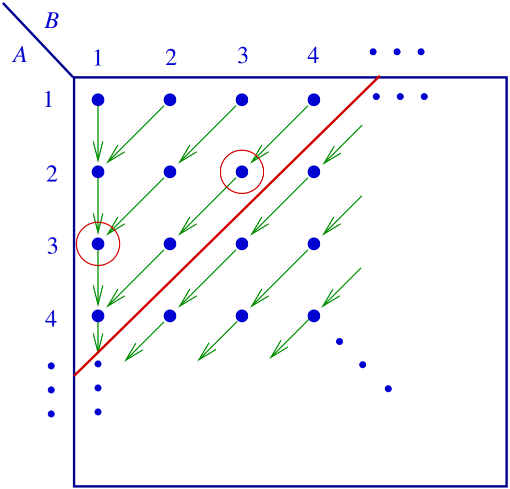

Residual Finiteness: Unary Algebras

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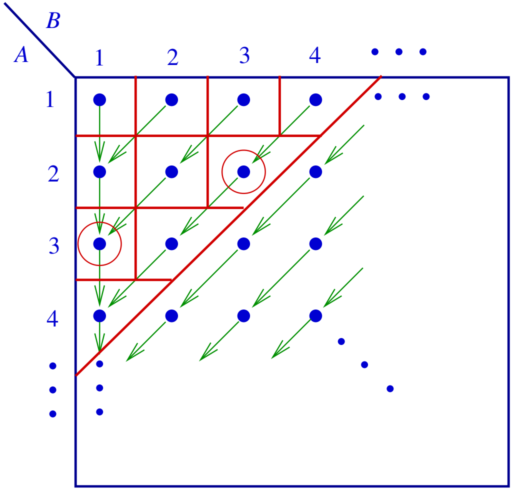

Some open problems

Open Problem

Find a necessary and sufficient condition for the wreath product $A \wr B$ of two monoids (or semigroups) to be residually finite.

Open Problem

Is the following problem algorithmically decidable: Given two finitely presented semigroups $S = \langle A \mid R \rangle$ and $T = \langle B \mid Q \rangle$, is their direct product finitely presented?

Open Problem

Is it true that $A \times B$ (A, B monoids, or even groups) is automatic if and only if G and H are automatic?