

One Relation Semigroups

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Statement of the Problem

Open Problem

Is the word problem soluble for every semigroup given by a single defining relation:

$$\langle a_1, \dots, a_n \mid u = v \rangle?$$



Presentations

$$\langle a_1, \dots, a_n \quad | \quad u_1 = v_1, \dots, u_m = v_m \rangle$$



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The semigroup S defined: the **largest/free-st semigroup generated by** (copies of) a_1, \dots, a_k , in which these generators satisfy all relations $u_j = v_j$ (and their consequences, but nothing else).
How to think about S : elements are words over $\{a_1, \dots, a_n\}$; some words are equal; two words are equal iff their equality is a consequence of the defining relations.



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Example

$S = \langle a, b \mid ba = a^2b \rangle$. Every word is equal to one of the form $a^i b^j$.



Word Problem

Definition

A semigroup S with a finite generating set A has a **soluble word problem** if there is an algorithm which for any two words $w_1, w_2 \in A^*$ decides whether or not they represent the same element of S .



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Example

$S = \langle a, b \mid ba = a^2b \rangle$. One can show:

$$a^i b^j = a^k b^l \text{ in } S \Leftrightarrow i = k \ \& \ j = l.$$

Algorithm for solving the word problem: Given two words w_1, w_2 transform them into $a^i b^j, a^k b^l$ and then test whether $i = k$ and $j = l$.



Brief Early History and Context

- ▶ 1900 – Hilbert's 10th Problem: *Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: to devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*



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- ▶ 1912 – Dehn: formulation of the word problem for groups
- ▶ 1931 – Gödel: incompleteness theorems for 1st order theories
- ▶ 1932 – Magnus: word problem for one-relator groups
- ▶ 1947 – Markov, Post: finitely presented semigroups with insoluble word problems
- ▶ 1951 – Markov: undecidability galore
- ▶ 195? – Novikov, Britton, Boone: finitely presented groups with insoluble word problems
- ▶ 1979 – Matiyasevich: negative solution to Hilbert's 10th Problem



Approaches

- ▶ Play with words (pages of induction 😞)
- ▶ Delegate (embeddings)
- ▶ Take apart (structure)
- ▶ Look at something else (other properties)

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Every group defined by a single relation has a soluble word problem.



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Theorem (Adyan 1966)

If u and v are non-empty words which have different first letters and different last letters then the semigroup defined by $\langle a_1, \dots, a_n \mid u = v \rangle$ embeds into the group with the same presentation, and hence has a soluble word problem.



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Remark

Some descendants:

- ▶ Diagrams (Remmers 1971, 1980) and pictures (Pride 1993)
- ▶ Small overlap semigroups (Remmers)
- ▶ Applications: Kashintsev, Guba, Howie, Pride, Jackson, . . .



Other Types of Semigroups

Theorem (Adjan, Oganessian 1987)

One relation problem can be reduced to presentations of the type:

$$\langle a, b \mid aua = avb, \rangle, \langle a, b \mid a = avb \rangle$$

Corollary

If every one relation right cancellative semigroup has a soluble word problem then every one relation semigroup has a soluble word problem.

Corollary (Ivanov, Margolis, Meakin 2001)

If every one relation inverse semigroup has a soluble word problem then every one relation semigroup has a soluble word problem.



Other Types of Semigroups

Theorem (Silva 1993)

One relation Clifford semigroups have a soluble word problem.

Question

How about completely regular semigroups?



Syntactical Approach: Special Monoids

Theorem (Adjan 1966)

Let S be the monoid defined by

$$\langle a_1, \dots, a_n \mid u = 1 \rangle.$$

The group of units is one relator (but not necessarily same presentation). The semigroup S has a soluble word problem.

Remark

See Zhang (1992) for a short proof and generalisation.



Structure: Some Speculations

Magnus's treatment of one relator groups: Freiheitssatz, 'large' subgroup, decompose into a product of free and/or 'smaller' one-relator groups.

Theorem (Semigroup Freiheitssatz; Squier, Wrathall 1983)

Let $S = \langle a_1, \dots, a_n \mid u = v \rangle$ be a one relation semigroup, and suppose that a_1 appears in u or v . Then the subsemigroup of S generated by $\{a_2, \dots, a_n\}$ is free.

Problem

- ▶ Investigate 'large' subsemigroups of one relation monoids.
- ▶ Candidates for large: $S \setminus \{a_1\}$; $S \setminus \langle a_1 \rangle$; ...
- ▶ Is there a natural decomposition?
- ▶ Do Rees index (Ruskuc 1998) or Green index (Gray, Ruskuc, to appear) help?



Other Properties

Investigate other structural, algebraic, combinatorial properties of one relation semigroups.

- ▶ Lallement 1974 – residual finiteness, idempotents
- ▶ Oganessian 1984 – isomorphism problem

A recent article:

A.J. Cain, V. Maltcev, Decision problems for finitely presented and one-relation semigroups and monoids, *Internat. J. Algebra Comput.*, to appear.



What if it isn't true?

Theorem (Matiyasevich 1967)

There exists a semigroup with three defining relations which has an insoluble word problem.

Theorem (Ivanov, Margolis, Meakin 2001)

Let S be the inverse monoid defined by $\langle A \mid u = 1 \rangle$, where w is a cyclically reduced word over $A \cup A^{-1}$. Let G be the group defined by the same presentation, and let P be the submonoid of G generated by all the prefixes of u . Then S has a soluble word problem if and only if the membership problem for P is soluble.



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