Residual Finiteness of Monoids, Associated Actions and Groups

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- *Closed under taking substructures (obvious).*
- *Closed under finite index extensions (in groups), finite Rees index extensions (in semigroups, NR, Thomas), direct products (in general).*
- *A finitely presented, r.f. algebraic structure has a soluble word problem (Mostowski 66, Evans 70).*

Residual Finiteness (3)

- W Magnus, Residually Finite Groups, 1969.
- D. Segal, Residually finite groups, 1990.
- A series of papers by Golubov et al., 1970s.

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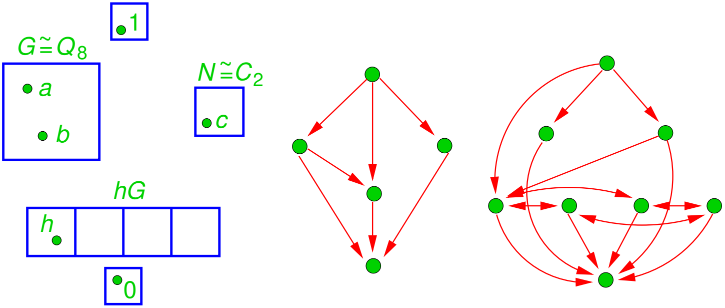
$\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.

Example: S, $S/\mathcal{H}$, $S/\mathcal{R}$, $S/\mathcal{L}$

$$S = \langle a, b, c, h \mid$$
$$aba = b,\ bab = a,\ c^3 = c,\ c^2h = h,\ ch = ha^2,$$
$$ac = bc = ca = cb = ah = bh = hc = h^2 = 0\rangle.$$

S, $S/\mathcal{H}$ $\qquad$ $S/\mathcal{R}$ $\qquad$ $S/\mathcal{L}$



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- $\Gamma_r(h) \cong \Gamma_l(h)$.
- $(h, h') \in \mathcal{R} \Rightarrow \Gamma_r(h) \cong \Gamma_l(h')$.

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Fact
A monoid is regular iff every $\mathcal{R}$-class contains an idempotent.

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☺ After all: S is finite iff $S/\mathcal{R}$, $S/\mathcal{L}$ and all $\Gamma_r(h)$ are finite and r.f. is a finiteness condition. ☺

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Theorem (Golubov 75)

Let S be a regular monoid of finite $\mathcal{J}$-type. Then S is residually finite if and only if all its maximal subgroups (i.e. Schützenberger groups) are residually finite.

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Proposition

If a regular monoid S is of finite $\mathcal{J}$-type then its action on its $\mathcal{R}$-classes is also residually finite.

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Hence, in T, $(\phi(e), \phi(f)) \notin \mathcal{L}$.
The action of S on the $\mathcal{L}$-classes of T is a finite homomorphic image of the action of S on $S/\mathcal{L}$ which separates L_s and L_t.

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This induces $\Gamma_r(h) \to \Gamma_r(\phi(h))$ which separates $\overline{s}$ and $\overline{t}$.

Summary (1)

Properties of M		M r.f.$\Rightarrow$ all $\Gamma(H)$ r.f.	M r.f. $\Rightarrow$ $M/\mathcal{L}$ r.f.	all $\Gamma(H)$ r.f. & $M/\mathcal{L}, M/\mathcal{R}$ r.f. $\Rightarrow$ M r.f.
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	regular			
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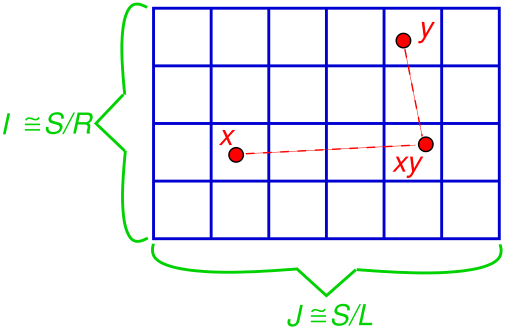
- two index sets I, J;
- $P = (p_{ji})_{j \in J, i \in I}$ – a $J \times I$ matrix with entries from G;

New semigroup: $S = M[G; I, J; P]$, on the set $I \times G \times J$, with multiplication

$$(i, g, j)(k, h, l) = (i, gp_{jk}h, l).$$

Rees Matrix Semigroups (2)



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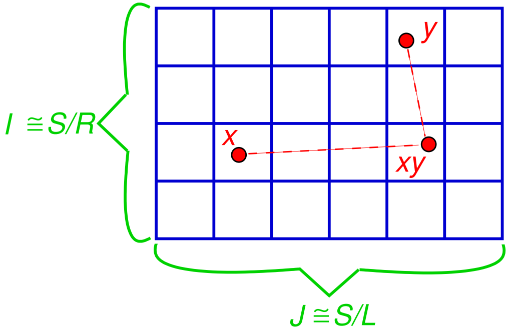

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- *S is regular.*

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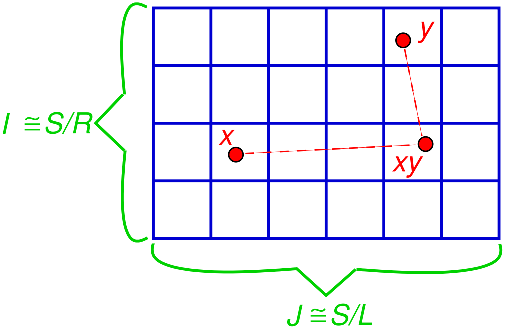

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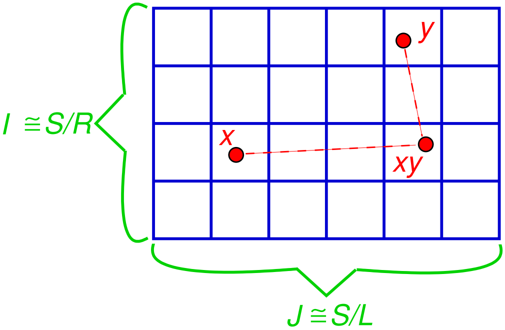

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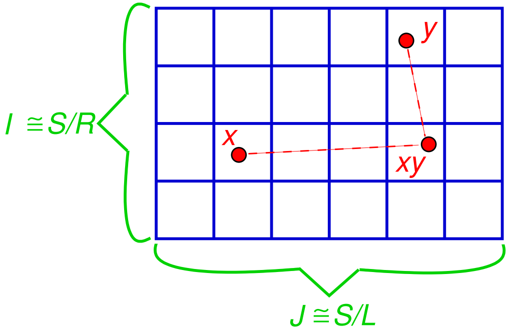

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- *Every element acts as a constant mapping on $S/\mathcal{L}$.*
- *The action of S on $S/\mathcal{L}$ is always residually finite.*

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Theorem (Golubov 72)

A Rees matrix semigroup $M[G; I, J; P]$ *is residually finite if and only if* G *is residually finite and* P *has only finitely many non-proportional rows and columns.*

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Theorem (Golubov 72)

A Rees matrix semigroup $M[G; I, J; P]$ is residually finite if and only if G is residually finite and P has only finitely many non-proportional rows and columns.

Corollary

There exists a non-residually finite regular semigroup in which the actions on $S/\mathcal{R}$ and $S/\mathcal{L}$, and all the maximal subgroups are residually finite.

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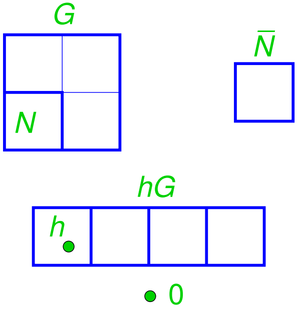
New semigroup:

$$\mathcal{S}(G, N) = \langle G, \overline{N}, h \,:\, hn = \overline{n}h,\ he_G = e_{\overline{N}}h = h,$$
$$g\overline{n} = \overline{n}g = gh = h\overline{n} = 0\ (g \in G,\ n \in N)\rangle.$$

Another Construction (2)



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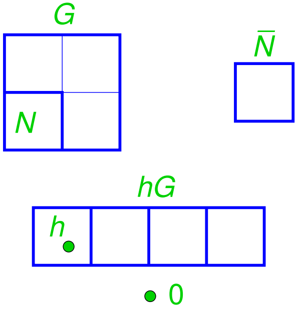
Facts

- *$S(G,N)$ is r.f. iff G is r.f.*



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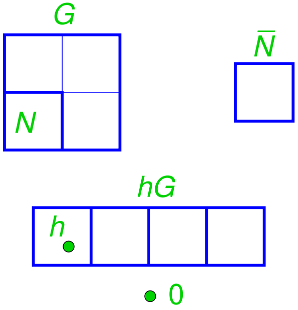
- *$S(G, N)$ is r.f. iff G is r.f.*
- *The action of $S(G, N)$ on its $\mathcal{L}$-classes is r.f. iff G/N is r.f.*



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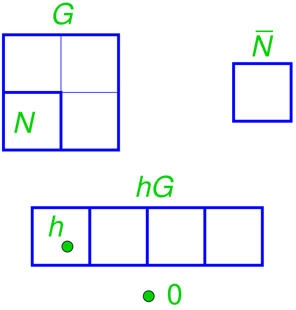

Corollary

There exists a residually finite semigroup such that its action on the $\mathcal{L}$-classes is not residually finite.

One More Counter-example (1)

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Let M be the commutative monoid with presentation

$$\langle a, a^{-1}, b_i, c_i, d, e \ (i \in \mathbb{Z}) \mid$$
$$aa^{-1} = a^{-1}a = 1, \ b_i c_i = d, \ b_i c_j = a^{\tau(j-i)} e \ (i \neq j),$$
$$b_i b_j = b_i c_j = b_i d = b_i e = c_j c_k = c_j d = c_j e = dd = de = ee = 0$$
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The $\mathcal{R}$-/$\mathcal{L}$-classes of M are:

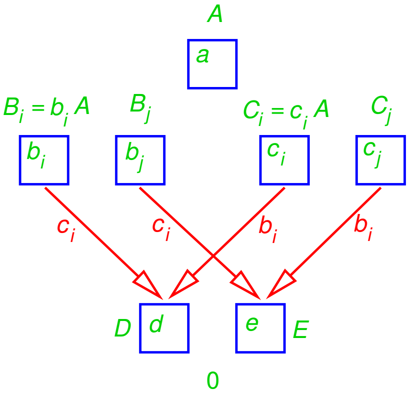
$A = \{a^{\pm p} \ : \ p \in \mathbb{Z}\}$ – the group of units

$B = \cup_{i \in \mathbb{Z}} B_i,\ B_i = Ab_i$

$C = \cup_{i \in \mathbb{Z}} C_i,\ C_i = Ac_i$

$D = Ad,\ E = Ae,\ \{0\}$.

One More Counter-example (2)



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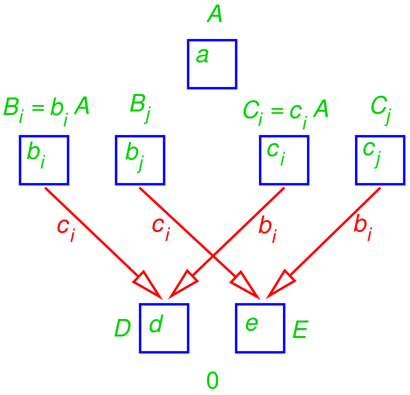

The action of M on $M/\mathcal{L}$ is not residually finite.

One More Counter-example (3)

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We still have one free choice: the function $\tau : \mathbb{Z} \setminus \{0\} \to \mathbb{Z}$.

One choice that works is:

$$\tau(2^k(2r+1)) = \frac{2}{3}(2^{2\lceil k/2 \rceil} - 1) \ (k, r \in \mathbb{Z}, \ k \geq 0)$$

Finitely Many Ideals

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Theorem

Let S be a monoid with finitely many left- and right ideals. Then S is residually finite if and only if all its Schützenberger groups are residually finite.

Summary (2)

Properties of M		M r.f.$\Rightarrow$ all $\Gamma(H)$ r.f.	M r.f. $\Rightarrow$ $M/\mathcal{L}$ r.f.	all $\Gamma(H)$ r.f. & $M/\mathcal{L}, M/\mathcal{R}$ r.f. $\Rightarrow$ M r.f.
general	arbitrary	✓		
	regular	✓	✓	
finite $\mathcal{J}$-type	arbitrary	✓		
	regular	✓	✓	✓
$< \infty$ many left/right ideals	arbitrary	✓	✓	
	regular	✓	✓	✓

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A Possible New Project

Definition

A an algebraic structure A is residually free if for any $x, y \in A$ with $x \neq y$ there exists a homomorphism f from A into a free object F such that $f(x) \neq f(y)$.

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Problem

Investigate residual freeness of semigroups and monoids. How does it depend on/affect residual freeness of its Schützenberger groups, and the actions on $\mathcal{R}$- and $\mathcal{L}$-classes?