# Residual Finiteness of Monoids, Associated Actions and Groups

Nik Ruskuc (joint work with Robert Gray)
nik@mcs.st-and.ac.uk

School of Mathematics and Statistics, University of St Andrews

Newcastle, 27 Nov 2008

University
of
St Andrews

# Residual Finiteness (1)

### Definition

An algebraic structure $A$ is residually finite if any, and hence all, of the following equivalent conditions hold:

- For all $x, y \in A$, $x \neq y$, there exists a homomorphism $f : A \to B$, $B$ finite, such that $f(x) \neq f(y)$.
- For all $x, y \in A$, $x \neq y$, there exists a finite index congruence $\rho$ which separates $x$ and $y$, i.e. $(x, y) \notin \rho$.
- The intersection of all finite index congruences of $A$ is trivial.

This applies to: groups, semigroups/monoids, actions, ...

# Residual Finiteness (2)

## Examples

- finite structures (r.f. is a finiteness condition);
- free – semigroups/ groups/ commutative semigroups/ abelian groups/inverse semigroups;
- infinite simple (congruence free) structures are not r.f.

## Facts

- *Closed under taking substructures (obvious).*
- *Closed under finite index extensions (in groups), finite Rees index extensions (in semigroups, NR, Thomas), direct products (in general).*
- *A finitely presented, r.f. algebraic structure has a soluble word problem (Mostowski 66, Evans 70).*

# Residual Finiteness (3)

- W Magnus, Residually Finite Groups, 1969.
- D. Segal, Residually finite groups, 1990.
- A series of papers by Golubov et al., 1970s.

# Right Regular Representation and Green's Equivalences $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$

Any monoid $S$ acts on itself via $(x, s) \mapsto xs$.

$\mathcal{R}$-classes := the strong orbits of this action.

$\mathcal{R}$ is a left congruence (i.e. $s \in S$ & $(x, y) \in \mathcal{R} \Rightarrow (sx, sy) \in \mathcal{R}$).

Hence, $S$ acts from the left on the set $S/\mathcal{R}$ of $\mathcal{R}$-classes.

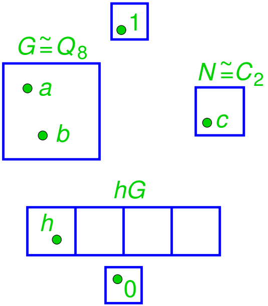
Left/right duality $\rightarrow$ $\mathcal{L}$-classes, right action on $S/\mathcal{L}$.

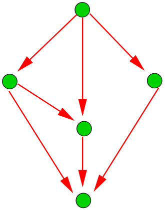
$\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.

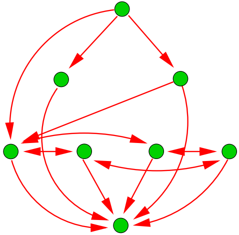
# Example: $S$, $S/\mathcal{H}$, $S/\mathcal{R}$, $S/\mathcal{L}$

$$S = \langle a, b, c, h \mid$$
$$aba = b,\ bab = a,\ c^3 = c,\ c^2h = h,\ ch = ha^2,$$
$$ac = bc = ca = cb = ah = bh = hc = h^2 = 0\rangle.$$

$S$, $S/\mathcal{H}$ $\quad$ $S/\mathcal{R}$ $\quad$ $S/\mathcal{L}$





# Schützenberger Groups (1)

Let $h \in S$, and let $H$ be the $\mathcal{H}$-class of $h$.

$\mathrm{Stab}_r(h) = \{s \in S \,:\, Hs = H\} \leq S$.

$s \sim t \Leftrightarrow hs = ht$ – a congruence on $\mathrm{Stab}_r(h)$.

$\Gamma_r(h) = \mathrm{Stab}_r(h)/\sim$.

$\Gamma_l(h)$ – defined dually.

# Schützenberger Groups (2)

## Facts

- $\Gamma_r(h)$ *is a group acting regularly on* $H$.
- $|\Gamma_r(h)| = |H|$.
- $\Gamma_r(h) \cong \Gamma_l(h)$.
- $(h, h') \in \mathcal{R} \Rightarrow \Gamma_r(h) \cong \Gamma_l(h')$.

# Idempotents, Subgroups, Regular Monoids

**Fact**

*If $h \in S$ is an idempotent then its $\mathcal{H}$-class $H$ is the largest subgroup of $S$ containing $h$ and $\Gamma_r(h) \cong H$.*

**Definition**

A monoid $S$ is regular if $(\forall x)(\exists y)(xyx = x)$.

**Fact**

*A monoid is regular iff every $\mathcal{R}$-class contains an idempotent.*

# In a Nutshell . . .

A monoid is ‘composed’ out of two actions (on $S/\mathcal{R}$ and $S/\mathcal{L}$) and a bunch of groups ($\Gamma_r(h)$, $h \in H$).

Is the following in any sense true:

$S$ is residually finite if and only if the actions on $S/\mathcal{R}$ and $S/\mathcal{L}$, and all the Schützenberger groups $\Gamma_r(h)$ ($h \in H$) are residually finite?

☺ After all: $S$ is finite iff $S/\mathcal{R}$, $S/\mathcal{L}$ and all $\Gamma_r(h)$ are finite and r.f. is a finiteness condition. ☺

# Positive Results: Regular Monoids (1)

### Definition

A monoid $S$ is of finite $\mathcal{J}$-type if every $\mathcal{J}$-class contains only finitely many $\mathcal{R}$- and $\mathcal{L}$-classes.

### Theorem (Golubov 75)

*Let $S$ be a regular monoid of finite $\mathcal{J}$-type. Then $S$ is residually finite if and only if all its maximal subgroups (i.e. Schützenberger groups) are residually finite.*

### Proposition

*If a regular monoid $S$ is of finite $\mathcal{J}$-type then its action on its $\mathcal{R}$-classes is also residually finite.*

# Positive Results: Regular Monoids (2)

### Theorem

*The action of a residually finite regular monoid on its $\mathcal{L}$-classes is residually finite.*

### Sketch of Proof

Let $L_s$, $L_t$ be two distinct $\mathcal{L}$-classes.
Let $e, f$ be idempotents s.t. $e \in L_s$, $f \in L_t$.
General theory: $(e, f) \notin \mathcal{L} \Leftrightarrow ef \neq e \vee fe \neq f$.
Wlog suppose $ef \neq e$.
Separate $e, ef$: $\phi : S \to T$, $T$ finite, $\phi(ef) \neq \phi(e)$.
Hence, in $T$, $(\phi(e), \phi(f)) \notin \mathcal{L}$.
The action of $S$ on the $\mathcal{L}$-classes of $T$ is a finite homomorphic image of the action of $S$ on $S/\mathcal{L}$ which separates $L_s$ and $L_t$.

# Positive Results: General Monoids

### Theorem

*If $S$ is a residually finite monoid then every Schützenberger group of $S$ is residually finite.*

### Sketch of Proof

(sketch) Let $\overline{s}$, $\overline{t}$ be distinct elements of $\Gamma_r(h)$.
That means that $hs \neq ht$ in $S$.
Separate $hs, ht$: $\phi : S \to T$, $T$ finite, $\phi(hs) \neq \phi(ht)$.
This induces $\Gamma_r(h) \to \Gamma_r(\phi(h))$ which separates $\overline{s}$ and $\overline{t}$.

# Summary (1)

| Properties of $M$ | | $M$ r.f.$\Rightarrow$ all $\Gamma(H)$ r.f. | $M$ r.f. $\Rightarrow$ $M/\mathcal{L}$ r.f. | all $\Gamma(H)$ r.f. & $M/\mathcal{L}, M/\mathcal{R}$ r.f. $\Rightarrow$ $M$ r.f. |
|---|---|---|---|---|
| general | arbitrary | ✓ | | |
| | regular | ✓ | ✓ | |
| finite $\mathcal{J}$-type | arbitrary | ✓ | | |
| | regular | ✓ | ✓ | ✓ |

# Rees Matrix Semigroups (1)

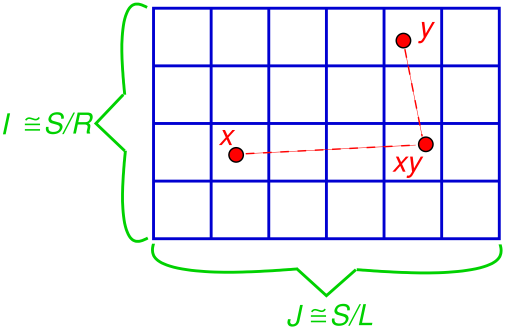
Ingredients:

- a group $G$;
- two index sets $I, J$;
- $P = (p_{ji})_{j\in J, i\in I}$ – a $J \times I$ matrix with entries from $G$;

New semigroup: $S = M[G; I, J; P]$, on the set $I \times G \times J$, with multiplication

$$(i, g, j)(k, h, l) = (i, gp_{jk}h, l).$$

# Rees Matrix Semigroups (2)




## Facts

- *$S$ is regular.*
- *$\mathcal{L}$-classes are indexed by the set $J$.*
- *Every element acts as a constant mapping on $S/\mathcal{L}$.*
- *The action of $S$ on $S/\mathcal{L}$ is always residually finite.*

# Rees Matrix Semigroups (3)

### Theorem (Golubov 72)

*A Rees matrix semigroup $M[G; I, J; P]$ is residually finite if and only if $G$ is residually finite and $P$ has only finitely many non-proportional rows and columns.*

### Corollary

*There exists a non-residually finite regular semigroup in which the actions on $S/\mathcal{R}$ and $S/\mathcal{L}$, and all the maximal subgroups are residually finite.*

# Another Construction (1)

Ingredients: A group $G$ and a normal subgroup $N \trianglelefteq G$.

Let $\overline{N} = \{\overline{n} \,:\, n \in N\}$ be a copy of $N$.

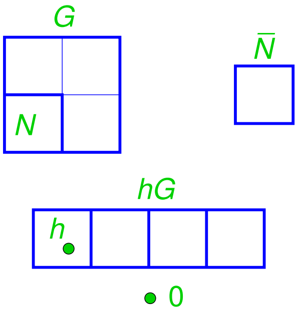
New semigroup:

$$\mathcal{S}(G,N) = \langle G, \overline{N}, h \,:\, hn = \overline{n}h,\ he_G = e_{\overline{N}}h = h,$$
$$g\overline{n} = \overline{n}g = gh = h\overline{n} = 0\ (g \in G,\ n \in N)\rangle.$$

# Another Construction (2)

## Facts

- *$S(G, N)$ is r.f. iff $G$ is r.f.*
- *The action of $S(G, N)$ on its $\mathcal{L}$-classes is r.f. iff $G/N$ is r.f.*




## Corollary

*There exists a residually finite semigroup such that its action on the $\mathcal{L}$-classes is not residually finite.*

# One More Counter-example (1)

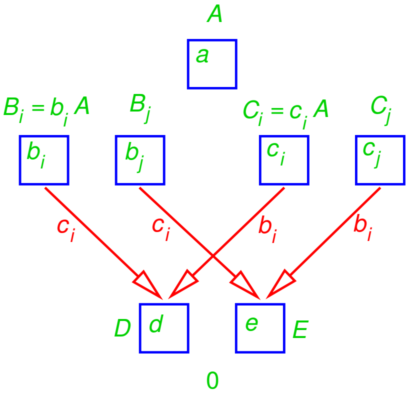
Let $M$ be the commutative monoid with presentation

$$\begin{gathered}\langle a, a^{-1}, b_i, c_i, d, e \ (i \in \mathbb{Z}) \mid \\ aa^{-1} = a^{-1}a = 1,\ b_i c_i = d,\ b_i c_j = a^{\tau(j-i)} e \ (i \neq j), \\ b_i b_j = b_i c_j = b_i d = b_i e = c_j c_k = c_j d = c_j e = dd = de = ee = 0 \\ (i, j, k \in \mathbb{Z})\rangle\end{gathered}$$

The $\mathcal{R}$-/$\mathcal{L}$-classes of $M$ are:

$$\begin{aligned} &A = \{a^{\pm p} \ : \ p \in \mathbb{Z}\} \text{ – the group of units} \\ &B = \cup_{i \in \mathbb{Z}} B_i,\ B_i = A b_i \\ &C = \cup_{i \in \mathbb{Z}} C_i,\ C_i = A c_i \\ &D = Ad,\ E = Ae,\ \{0\}. \end{aligned}$$

# One More Counter-example (2)




The action of $M$ on $M/\mathcal{L}$ is not residually finite.

# One More Counter-example (3)

We want $M$ to be residually finite.

We still have one free choice: the function $\tau : \mathbb{Z} \setminus \{0\} \to \mathbb{Z}$.

One choice that works is:

$$\tau(2^k(2r+1)) = \frac{2}{3}(2^{2\lceil k/2 \rceil} - 1) \ (k, r \in \mathbb{Z},\ \ k \geq 0)$$

# Finitely Many Ideals

Let's introduce a really strong finiteness condition on the actions on $S/\mathcal{R}$ and $S/\mathcal{L}$: finiteness itself.

🙂 That certainly guarantees that the two actions will be residually finite. 🙂

## Theorem

*Let $S$ be a monoid with finitely many left- and right ideals. Then $S$ is residually finite if and only if all its Schützenberger groups are residually finite.*

# Summary (2)

| Properties of $M$ | | $M$ r.f.$\Rightarrow$ all $\Gamma(H)$ r.f. | $M$ r.f. $\Rightarrow$ $M/\mathcal{L}$ r.f. | all $\Gamma(H)$ r.f. & $M/\mathcal{L}, M/\mathcal{R}$ r.f. $\Rightarrow$ $M$ r.f. |
|---|---|---|---|---|
| general | arbitrary | ✓ | ✗ | ✗ |
| | regular | ✓ | ✓ | ✗ |
| finite $\mathcal{J}$-type | arbitrary | ✓ | ✗ | ✗ |
| | regular | ✓ | ✓ | ✓ |
| $< \infty$ many left/right ideals | arbitrary | ✓ | ✓ | ✓ |
| | regular | ✓ | ✓ | ✓ |

# A Possible New Project

### Definition

A an algebraic structure $A$ is residually free if for any $x, y \in A$ with $x \neq y$ there exists a homomorphism $f$ from $A$ into a free object $F$ such that $f(x) \neq f(y)$.

### Problem

*Investigate residual freeness of semigroups and monoids. How does it depend on/affect residual freeness of its Schützenberger groups, and the actions on $\mathcal{R}$- and $\mathcal{L}$-classes?*