Diagonal Acts and Applications

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St Andrews

Problem E3311 Amer. Math. Monthly 96 (1989)

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Suppose S is a monoid containing elements a and b such that every element of $S \times S$ is of the form (au, bu) for some u in S (i.e., $S \times S$, considered as a right S-set, is cyclic).

(a) Show that S must be a singleton if S is any one of the following: finite, commutative, idempotent, or inverse.

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$$n\mu = \begin{cases} k\gamma & \text{if } n = 2k - 1 \\ k\delta & \text{if } n = 2k. \end{cases}$$

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Immediate check: $(\alpha\mu, \beta\mu) = (\gamma, \delta)$.

Diagonal Acts: Definitions

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Let S be a semigroup. The set S^n on which S acts via

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No infinite group has a finitely generated diagonal act.

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If $(a, b)s = (c, d)$ then $ab^{-1} = cd^{-1}$.

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Proposition (Gallagher 05)

No infinite inverse semigroup has a finitely generated diagonal act.

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Example (Robertson, NR, Thomson 01)

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Example (ibid)

There exists a finitely presented infinite monoid with cyclic diagonal acts.

Cyclic Diagonal Acts: Arities

Lemma

If $S \times S = (a, b)S$ *then* $S^{2^n} = (a_1, \ldots, a_{2^n})S$ *where* $\{a_1, \ldots, a_{2^n}\} = \{a, b\}^n$.

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Does S^2 finitely generated imply S^n finitely generated for all $n \geq 2$?

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- Ranks of direct powers (Neunhoeffer, Quick, NR).

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Let S be a semigroup. The finitary power semigroup of S (denoted $\mathcal{P}_f(S)$) consists of all finite subsets of S under multiplication $A \cdot B = \{ab \,:\, a \in A,\ b \in B\}$.

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Theorem

No, if S is a group (Gallagher, NR, 2007), or inverse semigroup with an infinite subgroup (Gallagher).

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Can $\mathcal{P}_f(S)$ be finitely presented for any infinite semigroup S?

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Problem

In the last case, is $\mathbf{d}(S)$ in fact eventually exponential?

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Growth: Infinite Semigroups

M. Neunhoeffer, M. Quick, NR, work in progress.

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Problem

If S is a finitely generated congruence-free semigroup, is $\mathbf{d}(S)$ eventually constant?