# Diagonal Acts and Applications


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# Problem E3311 Amer. Math. Monthly 96 (1989)

E 3311. *Proposed by Sydney Bulman-Fleming and Kenneth McDowell, Wilfrid Laurier University, Waterloo, Ontario.*

Suppose $S$ is a monoid containing elements $a$ and $b$ such that every element of $S \times S$ is of the form $(au, bu)$ for some $u$ in $S$ (i.e., $S \times S$, considered as a right $S$-set, is cyclic).

(a) Show that $S$ must be a singleton if $S$ is any one of the following: finite, commutative, idempotent, or inverse.

(b) Show that $S$ need not be a singleton in general.

## Proof of (b).

Take $S = T_{\mathbb{N}}$, the full transformation monoid.
Let $\alpha, \beta \in T_{\mathbb{N}}$ be defined by $n\alpha = 2n-1$, $n\beta = 2n$.
Let $\gamma, \delta \in T_{\mathbb{N}}$ be arbitrary.
Define $\mu \in T_{\mathbb{N}}$ by

$$n\mu = \begin{cases} k\gamma & \text{if } n = 2k-1 \\ k\delta & \text{if } n = 2k. \end{cases}$$

Immediate check: $(\alpha\mu, \beta\mu) = (\gamma, \delta)$.

# Diagonal Acts: Definitions

### Definition

Let $S$ be a semigroup. The set $S^n$ on which $S$ acts via

$$(x_1, \ldots, x_n)s = (x_1 s, \ldots, x_n s)$$

is called the ($n$-ary, right) diagonal act.

### Definition

The diagonal act $S^n$ is finitely generated if there is a finite set $A \subseteq S^n$ such that $S^n = AS$.

It is cyclic if $A$ can be chosen to have size 1.

# Diagonal Acts: Examples

### Example

If $S$ is any of $T_{\mathbb{N}}$, $P_{\mathbb{N}}$, $B_{\mathbb{N}}$ then the diagonal act $S^n$ is finitely generated for all $n$. (This includes $n = \aleph_0$!)

### Proposition

*No infinite group has a finitely generated diagonal act.*

### Proof

If $(a, b)s = (c, d)$ then $ab^{-1} = cd^{-1}$.

### Proposition (Gallagher 05)

*No infinite inverse semigroup has a finitely generated diagonal act.*

# Diagonal Acts: Examples

**Example (Robertson, NR, Thomson 01)**

The monoid of all recursive functions $\mathbb{N} \to \mathbb{N}$ is finitely generated and has cyclic diagonal acts.

**Example (ibid)**

There exists a finitely presented infinite monoid with cyclic diagonal acts.

# Cyclic Diagonal Acts: Arities

**Lemma**

*If* $S \times S = (a, b)S$ *then* $S^{2^n} = (a_1, \dots, a_{2^n})S$ *where* $\{a_1, \dots, a_{2^n}\} = \{a, b\}^n$.

**Sketch of Proof**

$(n = 2)$
$\{a, b\}^2 = \{aa, ba, ab, bb\}$.
Suppose we are given $(a_1, a_2, a_3, a_4) \in S^4$.
Find $s_1, s_2, s \in S$ so that:

$$(a, b)s_1 = (a_1, a_2),\ (a, b)s_2 = (a_3, a_4),\ (a, b)s = (s_1, s_2).$$

Then

$$(aa, ba, ab, bb)s = (as_1, bs_1, as_2, bs_2) = (a_1, a_2, a_3, a_4).$$

# Cyclic Diagonal Acts

### Corollary

*If the diagonal act $S^n$ is cyclic for some $n \geq 2$ then $S^n$ is cyclic for all $n \geq 2$.*

### Problem

*Does $S^2$ cyclic imply that $S^{\aleph_0}$ is cyclic?*

### Problem

*Does $S^2$ finitely generated imply $S^n$ finitely generated for all $n \geq 2$?*

# Applications

- Wreath products (Robertson, NR, Thomson);
- Finitary power semigroups (Robertson, Thomson, Gallagher, NR);
- Ranks of direct powers (Neunhoeffer, Quick, NR).

# Finitary Power Semigroups

**Definition**

Let $S$ be a semigroup. The finitary power semigroup of $S$ (denoted $\mathcal{P}_f(S)$) consists of all finite subsets of $S$ under multiplication $A \cdot B = \{ab \,:\, a \in A,\; b \in B\}$.

**Question**

Can $\mathcal{P}_f(S)$ be finitely generated for any infinite semigroup $S$?

**Theorem**

*No, if $S$ is a group (Gallagher, NR, 2007), or inverse semigroup with an infinite subgroup (Gallagher).*

# Finitary Power Semigroups

**Theorem**

*If $S$ is finitely generated and the diagonal act $S \times S$ is cyclic then $\mathcal{P}_f(S)$ is finitely generated.*

**Proof**

Suppose $S = \langle A \rangle$ and $S \times S = (a, b)S$.
Recall $S^{2^n} = (a_1, \ldots, a_{2^n})S$, where $\{a_1, \ldots, a_{2^n}\} = \{a, b\}^n$.
For $s \in S$, let $\overline{s} = \{s\}$; clearly $S \cong \overline{S} \leq \mathcal{P}_f(S)$.
$\mathcal{P}_f(S) = \langle \{a, b\} \rangle \overline{S} = \langle \{a, b\}, \overline{A} \rangle$.

**Problem**

*Can $\mathcal{P}_f(S)$ be finitely presented for any infinite semigroup $S$?*

# Growth Sequences of Direct Powers

Definition

$d(S)$ = the smallest number of generators needed to generate $S$.

Definition

$\mathbf{d}(S) = (d(S), d(S^2), d(S^3), \ldots)$.

Warning

From now on $S^n$ may stand for either the $n$-ary diagonal act of $S$ or for the $n$th direct power of $S$! In the above definition it is the latter.

Example

$\mathbf{d}(C_2) = (1, 2, 3, 4, 5, \ldots)$.

Example (Hall 1936)

$\mathbf{d}(A_5) = (2, 2, 2, \ldots, 2, 3,$

Example (Hall 1936)

$\mathbf{d}(A_5) = (\underbrace{2, 2, 2, \ldots 2}_{19}, 3, \ldots, 3, 4, \ldots)$

# Growth: Finite Groups and Semigroups

Wiegold and various co-authors, 1974–1995.

**Theorem**

*Let $G$ be a finite group.*

- *If $G$ is perfect then $\mathbf{d}(G)$ is logarithmic.*
- *If $G$ is non-perfect then $\mathbf{d}(G)$ is eventually linear.*

**Theorem**

*Let $S$ be a finite semigroup.*

- *If $S$ is a monoid then $\mathbf{d}(S)$ is eventually linear.*
- *If $S$ is not a monoid then $\mathbf{d}(S)$ is asymptotically exponential.*

**Problem**

*In the last case, is $\mathbf{d}(S)$ in fact eventually exponential?*

# Growth: Infinite Groups

Wiegold and various co-authors, 1974–1995.

**Theorem**

*Let $G$ be an infinite group.*

- *If $G$ is simple then $\mathbf{d}(G)$ is eventually constant.*
- *If $G$ is perfect, non-simple then $\mathbf{d}(G)$ is either logarithmic or eventually constant.*
- *If $G$ is non-perfect then $\mathbf{d}(G)$ is eventually linear.*

# Growth: Infinite Semigroups

M. Neunhoeffer, M. Quick, NR, work in progress.

## Theorem

*Let $S$ be a finitely generated semigroup with a cyclic diagonal act. Then $\mathbf{d}(S)$ is eventually constant.*

## Proof

Recall: d.a. $S^2$ cyclic $\Rightarrow$ d.a. $S^k$ cyclic for all $k$.
Suppose $S = \langle A \rangle$ and $S^k = (a_1, \ldots, a_k)S$.
For $s \in S$, let $\overline{s} = (s, \ldots, s) \in S^k$; clearly $\overline{S} \cong S$.
$S^k = (a_1, \ldots, a_k)\overline{S} = \langle (a_1, \ldots, a_k), \overline{A} \rangle$.
Hence $d(S^k) \leq d(S) + 1$.

## Problem

*If $S$ is a finitely generated congruence-free semigroup, is $\mathbf{d}(S)$ eventually constant?*