INVESTIGATING $p$-GROUPS BY COCLASS WITH GAP

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**Coclass Graph**

**Definition (Leedham-Green, Newman 1980)**

A finite $p$-group $G$ with $|G| = p^n$ and $\text{cl}(G) = c$ has coclass $cc(G) = n - c$. 

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**Introduction**

Computing coclass trees with Gap

Application: Cohomology of 2-groups

Application: $G(5, 1)$

Coclass 1
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**Notation:**

- If $GH$ edge, then $G$ is a descendant of $H$.  

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**Example: The graph** $\mathcal{G}(2, 1)$
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$2^2 \bullet \bullet$
Example: The graph $\mathcal{G}(2, 1)$

$2^2 \quad V_4$
Example: The graph $G(2, 1)$

$$
\begin{array}{ccc}
2^2 & V_4 & C_4 \\
\bullet & \bullet & \bullet
\end{array}
$$
**Example: The graph \( G(2, 1) \)**

\[
\begin{array}{ccc}
2^2 & V_4 & C_4 \\
2^3 & & \\
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\]
**EXAMPLE: THE GRAPH \( G(2, 1) \)**

\[
\begin{array}{ccc}
2^2 & V_4 & C_4 \\
2^3 & D_8 & \\
\end{array}
\]
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\[
\begin{array}{ccc}
2^2 & V_4 & C_4 \\
2^3 & D_8 & Q_8 \\
\end{array}
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\[
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\]
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```
\begin{center}
\begin{tikzpicture}
  \node (V4) at (0,1) {$V_4$};
  \node (C4) at (1,1) {$C_4$};
  \node (D8) at (0,0) {$D_8$};
  \node (Q8) at (1,0) {$Q_8$};
  \draw (V4) -- (D8);
  \draw (C4) -- (Q8);
\end{tikzpicture}
\end{center}
```
Example: The graph $\mathcal{G}(2, 1)$

\[
\begin{array}{c}
2^2 & \bullet & V_4 & \bullet & C_4 \\
2^3 & \bullet & D_8 & \bullet & Q_8 \\
2^4 & \bullet & \bullet & \bullet & D_{16} & \bullet & Q_{16}
\end{array}
\]
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```
2^2

V_4

C_4

2^3

D_8

Q_8

2^4

D_{16}

Q_{16}

SD_{16}
```
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```
2^2
  V_4
/   |
D_8  Q_8
/   |
D_{16} Q_{16} SD_{16}
```

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Coclass 1
EXAMPLE: THE GRAPH $G(2, 1)$
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The graph $G(2, 1)$ is a cohomology of 2-groups. It shows the structure of cohomology classes for each power of 2, with groups such as $V_4$, $C_4$, $D_8$, $Q_8$, $D_{16}$, $Q_{16}$, $SD_{16}$, $D_{32}$, $Q_{32}$, $SD_{32}$, $D_{64}$, $Q_{64}$, and $SD_{64}$.
EXAMPLE: THE GRAPH $G(2, 1)$
**Main Conjecture**

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Let $r \in \mathbb{N}$. The $p$-groups of coclass $r$ can be split into finitely many *coclass families*. 

---

*Investigating $p$-groups by coclass with Gap*

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  - *Schur multiplicators*

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  - Schur multiplicators,
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  - cohomology rings $H^*(-, R)$, for $R$ ring,

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Let $G \in \mathcal{G}(p, r)$.

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- The \( i \)-th branch \( \mathcal{B}_i \) of \( \mathcal{T}_G \) is generated by all descendants of \( G_i \) which are not descendants of \( G_{i+1} \).
- Depth of \( \mathcal{B}_i \) = length of a longest path in \( \mathcal{B}_i \).
- Width of \( \mathcal{B}_i \) = maximum number of groups of same order in \( \mathcal{B}_i \).
Example: The graph $G(2, 1)$ revisited
**Conjecture**

If \( T \) is a coclass tree \( G(p, r) \), then there exist \( d, f \in \mathbb{N} \) such that:

- \( B_{i+d} \) can be constructed from \( B_i \) for \( i \geq f \).
CONSTRUCTION RULES AND COCLASS FAMILIES

CONJECTURE

\( T \) coclass tree \( G(p, r) \). \( \Rightarrow \) Exist \( d, f \in \mathbb{N} \) such that

- \( B_{i+d} \) can be constructed from \( B_i \) for \( i \geq f \);
- we get a surjective map \( \varphi_i : B_{i+d} \rightarrow B_i \) for \( i \geq f \).
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Application: $G(5, 1)$

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Choose suitable $m \geq f$ and $G \in B_i$ with $m \leq i < m + d$. 
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Choose suitable \( m \geq f \) and \( G \in B_i \) with \( m \leq i < m + d \). Then \( G \) defines an infinite coclass family \( \mathcal{F}_G \) consisting of \( G \) and iterated preimages of \( G \) under \( \varphi_{i+dj} \), for \( j \in \mathbb{N} \).
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Construction Rules and Coclass Families

Conjecture

$T$ coclass tree $G(p, r) \Rightarrow$ Exist $d, f \in \mathbb{N}$ such that

- $\mathcal{B}_{i+d}$ can be constructed from $\mathcal{B}_i$ for $i \geq f$;
- we get a surjective map $\varphi_i : \mathcal{B}_{i+d} \to \mathcal{B}_i$ for $i \geq f$.

Choose suitable $m \geq f$ and $G \in \mathcal{B}_i$ with $m \leq i < m + d$.
Then $G$ defines an infinite coclass family $\mathcal{F}_G$ consisting of $G$ and iterated preimages of $G$ under $\varphi_{i+dj}$, for $j \in \mathbb{N}$.

Theorem (Eick, Leedham-Green)

Let $T$ be a bounded coclass tree. Then there exist $d, f \in \mathbb{N}$ and isomorphisms $\mathcal{B}_{i+d} \to \mathcal{B}_i$, $i \geq f$. 
EXAMPLE: $\mathcal{G}(2, 1)$ AND $\mathcal{G}(2, 2)$

$\mathcal{G}(2, 1)$ contains 6 coclass families.
**Example: $\mathcal{G}(2, 1)$ and $\mathcal{G}(2, 2)$**

$\mathcal{G}(2, 1)$ contains 6 coclass families:

- 3 finite families (containing $C_4$, $V_4$ and $Q_8$, resp.).
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\( \mathcal{G}(2, 1) \) contains 6 coclass families:
- 3 finite families (containing \( C_4 \), \( V_4 \) and \( Q_8 \), resp.) and
- 3 infinite families (dihedral, quaternion, semi-dihedral).
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$\mathcal{G}(2,2)$ contains
- 5 coclass trees $\mathcal{T}_1(2,2), \ldots, \mathcal{T}_5(2,2)$. 
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**Theorem**

There is a 1–1 correspondence between the pro-$p$-groups of coclass $r$ and the mainlines of coclass trees in $\mathcal{G}(p, r)$. 

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Coclass 1
**Theorem**

There is a 1–1 correspondence between the pro-$p$-groups of coclass $r$ and the mainlines of coclass trees in $G(p, r)$.

**Definition**

A uniserial $p$-adic space group of dimension $d$ is an extension.
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**Definition**

A uniserial $p$-adic space group of dimension $d$ is an extension of

- a **translation subgroup** $T = \mathbb{Z}_p^d$ ($\mathbb{Z}_p$ $p$-adic integers).
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Number of uniserial $p$-adic space groups:

$p = 2$: Newman, O’Brien

$p$ odd: construction algorithm by Eick
**Number of pro-\( p \)-groups**

Number of uniserial \( p \)-adic space groups:
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APPLICATION: $G(5, 1)$

Coclass 1

Number of pro-$p$-groups:

- 5 pro-$2$-groups of coclass 2,
- 54 pro-$2$-groups of coclass 3.

Number of uniserial $p$-adic space groups:

$p = 2$: Newman, O’Brien

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Number of uniserial $p$-adic space groups:

$p = 2$: Newman, O’Brien

$p$ odd: construction algorithm by Eick
**Input:** $p$-group $G$ of coclass $r$
Computing Immediate Descendants

**Input:** $p$-group $G$ of coclass $r$

**Output:** Immediate descendants of $G$

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Coclass 1
Computing Immediate Descendants

Input: $p$-group $G$ of coclass $r$
Output: Immediate descendants of $G$
(i.e. all central extensions of $C_p$ by $G$ of coclass $r$)
**Computing Immediate Descendants**

**Input:** $p$-group $G$ of coclass $r$

**Output:** Immediate descendants of $G$ (i.e. all central extensions of $C_p$ by $G$ of coclass $r$)

**Sketch of algorithm:**

- Let $U \subseteq Z^2(G, C_p)$ correspond to the central extensions of coclass $r$. 

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**Computing Immediate Descendants**

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**Output:** Immediate descendants of $G$  
(i.e. all central extensions of $C_p$ by $G$ of coclass $r$)

**Sketch of algorithm:**

- Let $U \subseteq Z^2(G, C_p)$ correspond to the central extensions of coclass $r$.
- Consider $U \subseteq \mathbb{F}_p^I$ for some $I$. 

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Heiko Dietrich*, Bettina Eick, Dörte Feichtenschlager*
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**Theorem:** There is a 1-1 correspondence between the $\text{Aut}(G)$-orbits of $U$ and the isomorphism types of immediate descendants of $G$. 
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**Theorem**

*There is a 1–1 corresp. between the \( \text{Aut}(G) \)-orbits of \( \overline{U} \) and the isomorphism types of immediate descendants of \( G \).*
**Conjecture for cohomology**

Let $k$ be a field with $\text{char}(k) = 2$ and $r \in \mathbb{N}$. Then there exist only finitely many isomorphism types of cohomology rings $H^*(G, k)$ where $G$ is a 2-group of coclass $r$. 

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**Introduction**

Computing coclass trees with Gap

Application: Cohomology of 2-groups

Application: $G(5, 1)$

Coclass 1
**Conjecture for Cohomology**

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Let $k$ be a field with $\text{char}(k) = 2$ and $r \in \mathbb{N}$. Then there exist only finitely many isomorphism types of cohomology rings $H^*(G, k)$ where $G$ is a 2-group of coclass $r$.

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Let $k$ be a field with $\text{char}(k) = 2$ and $\mathcal{F}_G$ a coclass family of 2-groups.
**THEOREM (CARSLON)**

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**CONJECTURE**

Let $k$ be a field with $\text{char}(k) = 2$ and $\mathcal{F}_G$ a coclass family of 2-groups. Then $H^*(G, k) \cong H^*(H, k)$ for all $H \in \mathcal{F}_G$. 
THEOREM

If $i, j \geq 4$, then

- $H^*(D_{2i}, \mathbb{F}_2) \cong H^*(D_{2j}, \mathbb{F}_2)$. 
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If \( i, j \geq 4 \), then

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Furthermore,

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Conjectured mod2-cohomology for $B_{2i}, B_{2i+1} \subseteq T_1(2, 2)$ ($i \in \mathbb{N}_0$).
\( \mathcal{G}(2, 2) \) AND COHOMOLOGY: \( \mathcal{T}_2(2, 2) \)

mod2-cohomology for \( \mathcal{B}_i \subseteq \mathcal{T}_2(2, 2) \) (0 \( \leq i \leq 4 \)).
Conjectured mod2-cohomology for $B_i \subseteq \mathcal{T}_3(2, 2)$ ($i \in \mathbb{N}_0$).
$G(2,2)$ AND COHOMOLOGY: $\mathcal{T}_4(2,2)$

Conjectured mod2-cohomology for $B_i \subseteq \mathcal{T}_4(2,2)$ ($i \in \mathbb{N}_0$).
\( G(2, 2) \) AND COHOMOLOGY: \( \mathcal{T}_5(2, 2) \)

Conjectured mod2-cohomology for \( \mathcal{B}_i \subseteq \mathcal{T}_5(2, 2) \) \((i \in \mathbb{N})\).
Widths and Depths

We have seen that $G(2, r)$, $r \geq 1$, and $G(3, 1)$ are bounded width and depth, and we expect that $G(5, 1)$: bounded width, unbounded depth. $G(p, 1)$, $p \geq 7$: unbounded width and depth. $G(p, r)$, $p \geq 3$, $r \geq 2$: complex structure.

Consider $G(5, 1)$ in more detail.
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It is known:

- Pro-$p$-group of coclass 1 is $S = C_p \rtimes T$ with $T = \mathbb{Z}_p^{p-1}$. 
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- Let the \textit{collar} $B_i(l, k)$ be defined as $B_i(k) \setminus B_i(l - 1)$. 
THE BRANCH $B_i$ OF $G(5, 1)$

**Conjecture**

Let $i \geq 8$ and write $i = 8 + 4x + y$ with $0 \leq y \leq 3$ and $x \geq 0$. 
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\[ \text{H}(i) \sim = \text{H}(i + 4) \]
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**The branch \( \mathcal{B}_i \) of \( \mathcal{G}(5, 1) \)**

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- \( H(i) \cong H(i + 4) \) and \( T(i) \cong T(i + 4) \).
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THE BRANCHES OF $\mathcal{G}(5, 1)$

Structures of $B_i, B_{i+4}, \ldots$ with $12 \leq i \leq 15$. 
The conjectured branches $B_i$ with $i = 8 + 4x + 1$ and $x \geq 0$. 
The origins of infinite coclass families in $\mathcal{B}_i$, $12 \leq i \leq 15$. 
The origins of infinite coclass families in $\mathcal{B}_i$, $12 \leq i \leq 15$. 
$G(5, 1)$ AND COCLASS FAMILIES

The origins of infinite coclass families in $B_i$, $12 \leq i \leq 15$. 

\[ B_i, B_{i+4}, B_{i+8} \]
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\[ \mathcal{B}_i \]

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$G(5, 1)$ AND COCLASS FAMILIES

The origins of infinite coclass families in $B_i$, $12 \leq i \leq 15$. 
\textbf{\(G(5, 1)\) AND COCLASS FAMILIES}

\textbf{Infinite coclass families in \(G(5, 1)\):

- The groups in \(H(i), T(i),\) and \(C(i, 0)\) with \(12 \leq i \leq 15\)
  would define disjoint infinite coclass families.}
\( G(5, 1) \) AND COCLASS FAMILIES

Infinite coclass families in \( G(5, 1) \):

- The groups in \( H(i) \), \( T(i) \), and \( C(i, 0) \) with \( 12 \leq i \leq 15 \) would define disjoint infinite coclass families.
- Their union would contain all groups in \( G(5, 1) \) which are descendants of \( G_{12} \).
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Infinite coclass families in $G(5, 1)$:
- The groups in $H(i)$, $T(i)$, and $C(i, 0)$ with $12 \leq i \leq 15$ would define disjoint infinite coclass families.
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Conjectured number of infinite coclass families:

<table>
<thead>
<tr>
<th># families</th>
<th>$y = 0$</th>
<th>$y = 1$</th>
<th>$y = 2$</th>
<th>$y = 3$</th>
<th>$\Sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>in the heads</td>
<td>366</td>
<td>578</td>
<td>741</td>
<td>953</td>
<td>2638</td>
</tr>
<tr>
<td>in the collars</td>
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<td>756</td>
<td>748</td>
<td>756</td>
<td>3008</td>
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<tr>
<td>in the tails</td>
<td>730</td>
<td>735</td>
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<td>737</td>
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<tr>
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<td>1844</td>
<td>2069</td>
<td>2219</td>
<td>2446</td>
<td>8578</td>
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</table>
**Conjecture**

For $n \in \mathbb{N}$ write $n = 4s_n + r_n$ with $1 \leq r_n \leq 4$. 
CONJECTURE

For $n \in \mathbb{N}$ write $n = 4s_n + r_n$ with $1 \leq r_n \leq 4$.

- If $G \in \mathcal{B}_i$ with $i \geq 8$ is capable and $|G| = p^n$, then

$$I(M(G)) = \begin{cases} (5, \; 5^{s_n}, \; 5^{s_n}) & \text{if } r_n = 1, 2, \\ (5, \; 5^{s_n}, \; 5^{s_n+1}) & \text{if } r_n = 3, 4. \end{cases}$$
**CONJECTURE**

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  \end{cases}$$

- If $H$ is a *terminal immediate descendant* of $G$, then

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  \end{cases}$$
$G(5, 1)$ AND SCHUR MULTIPLICATORS

**Coclass families:**

- Let $G \in \mathcal{B}_i$ define a coclass family $\mathcal{F}$. 

*Introduction*

**Computing coclass trees with GAP**

**Application:** Cohomology of 2-groups

**Application:** $G(5, 1)$

**Coclass 1**
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- Let $G \in \mathcal{B}_i$ define a coclass family $\mathcal{F}$.
- Let $K \in \mathcal{F}$ and $I(M(G)) = (5^a, 5^b, 5^c)$, $a \in \{0, 1\}$. 
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Then:

$K \in B_j$ for some $j$ with $j - i = 4l$, $l \in \mathbb{N}_0$, and we would have:
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**G(5, 1) AND SCHUR MULTIPLICATORS**

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\[
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\end{cases}
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(5^a, 5^{b+2l}, 5^{c+2l}) & \text{if } G \in T(i), \\
(5^a, 5^{b+l+k}, 5^{c+l+k}) & \text{if } G \in C(i, 0), K \in C(j, k).
\end{cases}$$
Conjecture

Let $\mathcal{F}$ be an infinite coclass family. Let $G \in \mathcal{F} \cap \mathcal{B}_i$ with $i \geq 12$. 
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Moreover,

- if $\mathcal{F}$ arises from a head, then $u \in \{1, 2, 4, 16\}$ and $v \in \{-1, 0, 1, 2, 3\}$.
- if $\mathcal{F}$ arises from a tail or collar, then $u \in \{1, 2, 4\}$ and $v \in \{2, 3\}$.  

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**$G(5, 1)$ and outer automorphism groups**
Let $p > 3$ be a prime.

Recall: $S = C_p \rtimes T$ and $G_i = S/\gamma_{i+2}(S)$. 
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  $\text{hom} \cong A^{(p-3)/2}$. 
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- For $\gamma \in H^2(G_i, A)$ let $E(\gamma)$ be the corresp. extension.
- Fix $\varepsilon \in \text{ext}$ with $E(\varepsilon)$ of coclass 1 ($\sim\sim$ mainline).
- $H \in \overline{B}_i \iff \varepsilon + \tau + \kappa$ where $\tau \in \text{twig}$ and $\kappa \in \text{hom}$ with $\kappa \notin H^2(G_i, \gamma_2(S)/\gamma_j(S))$. 

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$\Rightarrow$ Compute $\Sigma$-orbits.
So far, this theory leads to...
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- The isomorphism $H^2(G_i, A) \rightarrow H^2(G_{i+p-1}, A)$ is a $\Sigma$-module isomorphism.
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**Construct groups of coclass 1**

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ضبط الخطوة الأولى لثبت التخمين الرئيسي للجماعات من الكلاس 1.
Thank you for your kind attention.