GradedModules
A homalg based package for the Abelian category of finitely presented graded modules over a computable graded ring
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(this manual is still under construction)

This manual is best viewed as an HTML document. The latest version is available ONLINE at:

http://homalg.math.rwth-aachen.de/~markus/GradedModules/chap0.html

An OFFLINE version should be included in the documentation subfolder of the package. This package is part of the homalg-project:

http://homalg.math.rwth-aachen.de/index.php/unreleased/gradedmodules
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Introduction
Chapter 2

Installation of the \textbf{GradedModules Package}

To install this package just extract the package’s archive file to the GAP pkg directory.

By default the \texttt{GradedModules} package is not automatically loaded by GAP when it is installed. You must load the package with

\begin{verbatim}
LoadPackage("GradedModules");
\end{verbatim}

before its functions become available.

Please, send me an e-mail if you have any questions, remarks, suggestions, etc. concerning this package. Also, I would be pleased to hear about applications of this package.

Mohamed Barakat
Chapter 3

Quick Start
Chapter 4

Ring Maps

4.1 Ring Maps: Attributes

4.1.1 KernelSubobject

\[ \text{KernelSubobject}(\phi) \]  
\[ \text{(method)} \]

\textbf{Returns:} a homalg submodule
The kernel ideal of the ring map \( \phi \).

4.2 Ring Maps: Operations and Functions

4.2.1 SegreMap

\[ \text{SegreMap}(R, s) \]  
\[ \text{(method)} \]

\textbf{Returns:} a homalg ring map
The ring map corresponding to the Segre embedding of \( \text{MultiProj}(R) \) into the projective space according to \( P(W_1) \times P(W_2) \to P(W_1 \otimes W_2) \).

4.2.2 PlueckerMap

\[ \text{PlueckerMap}(l, n, A, s) \]  
\[ \text{(method)} \]

\textbf{Returns:} a homalg ring map
The ring map corresponding to the Plücker embedding of the Grassmannian \( G_l(P^n(A)) = G_l(P(W)) \) into the projective space \( P(\wedge^l W) \), where \( W = V^* \) is the \( A \)-dual of the free module \( V = A^{n+1} \) of rank \( n + 1 \).

4.2.3 VeroneseMap

\[ \text{VeroneseMap}(n, d, A, s) \]  
\[ \text{(method)} \]

\textbf{Returns:} a homalg ring map
The ring map corresponding to the Veronese embedding of the projective space \( P^n(A) = P(W) \) into the projective space \( P(S^d W) \), where \( W = V^* \) is the \( A \)-dual of the free module \( V = A^{n+1} \) of rank \( n + 1 \).
Chapter 5

GradedModules

5.1 GradedModules: Category and Representations

5.2 GradedModules: Constructors

5.3 GradedModules: Properties

For more properties see the corresponding section (Modules: Modules: Properties) in the documentation of the homalg package.

5.4 GradedModules: Attributes

5.4.1 BettiTable (for modules)

\[ \text{BettiTable}(M) \] (attribute)

Returns: a homalg diagram

The Betti diagram of the homalg graded module \(M\).

5.4.2 CastelnuovoMumfordRegularity

\[ \text{CastelnuovoMumfordRegularity}(M) \] (attribute)

Returns: an integer

The Castelnuovo-Mumford regularity of the homalg graded module \(M\).

5.4.3 CastelnuovoMumfordRegularityOfSheafification

\[ \text{CastelnuovoMumfordRegularityOfSheafification}(M) \] (attribute)

Returns: an integer

The Castelnuovo-Mumford regularity of the sheafification of homalg graded module \(M\).

For more attributes see the corresponding section (Modules: Modules: Attributes) in the documentation of the homalg package.
5.5 LISHV: Logical Implications for GradedModules

5.6 GradedModules: Operations and Functions

5.6.1 MonomialMap

\[ \text{MonomialMap}(d, M) \]

(definition, operation)

\textbf{Returns:} a homalg map

The map from a free graded module onto all degree \( d \) monomial generators of the finitely generated homalg module \( M \).

\[ \text{Example} \]

\begin{verbatim}
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> S := GradedRing( R );;
gap> M := HomalgMatrix( "[ x^3, y^2, z, z, 0, 0 ]", 2, 3, S );;
gap> M := LeftPresentationWithDegrees( M, [ -1, 0, 1 ] );
< A graded non-torsion left module presented by 2 relations for 3 generators>
gap> m := MonomialMap( 1, M );
< A homomorphism of graded left modules>
gap> Display( m );
\end{verbatim}

\textit{the graded map is currently represented by the above 10 x 3 matrix}

(degrees of generators of target: [ -1, 0, 1 ])

5.6.2 RandomMatrix

\[ \text{RandomMatrix}(S, T) \]

(definition, operation)

\textbf{Returns:} a homalg matrix

A random matrix between the graded source module \( S \) and the graded target module \( T \).

\[ \text{Example} \]

\begin{verbatim}
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "a,b,c";;
gap> S := GradedRing( R );;
gap> rand := RandomMatrix( S^1 + S^2, S^2 + S^3 + S^4 );
<A 2 x 3 matrix over a graded ring>
gap> Display( rand );
\end{verbatim}

#-3*a-b, -1,
#-a^2+2*a*b+2*b^2-2*a*c+2*b*c+c^2, -a+c,
#-2*a^3+5*a^2*b-3*b^3+3*a*b*c+3*b^2*c+2*a*c^2+2*b*c^2+c^3-3*b^2-2*a*c-2*b*c+c^2
5.6.3 GeneratorsOfHomogeneousPart

GeneratorsOfHomogeneousPart(d, M)  
(operation)

Returns: a homalg matrix

The resulting homalg matrix consists of a generating set (over \(R\)) of the \(d\)-th homogeneous part of the finitely generated homalg \(S\)-module \(M\), where \(R\) is the coefficients ring of the graded ring \(S\) with \(S_0 = R\).

Example

```gap
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> S := GradedRing( R );;
gap> M := HomalgMatrix( "[ x^3, y^2, z, z, 0, 0 ]", 2, 3, S );;
gap> M := LeftPresentationWithDegrees( M, [ -1, 0, 1 ] );;
<An unevaluated non-zero 7 x 3 matrix over a graded ring>
gap> m := GeneratorsOfHomogeneousPart( 1, M );
<An unevaluated non-zero 7 x 3 matrix over a graded ring>
gap> Display( m );
x^2,0,0,
x*y,0,0,
y^2,0,0,
0, x,0,
0, y,0,
0, z,0,
0, 0,1
(over a graded ring)
```

Compare with MonomialMap (5.6.1).

5.6.4 SubmoduleGeneratedByHomogeneousPart

SubmoduleGeneratedByHomogeneousPart(d, M)  
(operation)

Returns: a homalg module

The submodule of the homalg module \(M\) generated by the image of the \(d\)-th monomial map (→ MonomialMap (5.6.1)), or equivalently, by the generating set of the \(d\)-th homogeneous part of \(M\).

Example

```gap
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> S := GradedRing( R );;
gap> M := HomalgMatrix( "[ x^3, y^2, z, z, 0, 0 ]", 2, 3, S );;
gap> M := LeftPresentationWithDegrees( M, [ -1, 0, 1 ] );
<An unevaluated non-zero 7 x 3 matrix over a graded ring>
gap> n := SubmoduleGeneratedByHomogeneousPart( 1, M );
<An graded left submodule given by 7 generators>
gap> Display( M );
z, 0, 0,
0, y^2*z,z^2,
x^3,y^2, z
Cokernel of the map
Q[x,y,z]^(1x3) --> Q[x,y,z]^(1x3),
currently represented by the above matrix
(graded, degrees of generators: [ -1, 0, 1 ])
```
GradedModules

\begin{verbatim}
gap> Display( n );
x^2,0,0,
x*y,0,0,
y^2,0,0,
0, x,0,
0, y,0,
0, z,0,
0, 0,1

A left submodule generated by the 7 rows of the above matrix

(graded, degrees of generators: [ 1, 1, 1, 1, 1, 1, 1 ])
gap> N := UnderlyingObject( n );
<A graded left module presented by yet unknown relations for 7 generators>
gap> Display( N );
z, 0, 0,0, 0, 0,0,
0, z, 0,0, 0, 0,0,
0, 0, z,0, 0, 0,0,
0, 0, 0,0, -z, y,0,
x, 0, 0,0, y, 0,z,
-y,x, 0,0, 0, 0,0,
0, -y,x,0, 0, 0,0,
0, 0, 0,-y, x, 0,0,
0, 0, 0,-z, 0, x,0,
0, 0, 0,0, y*z,0,z^2,
0, 0, 0,y^2*z,0, 0,x*z^2

Cokernel of the map

\mathbb{Q}[x,y,z]^{(1x11)} \rightarrow \mathbb{Q}[x,y,z]^{(1x7)},

currently represented by the above matrix

(graded, degrees of generators: [ 1, 1, 1, 1, 1, 1, 1 ])
gap> gens := GeneratorsOfModule( N );
<A set of 7 generators of a homalg left module>
gap> Display( gens );
x^2,0,0,
x*y,0,0,
y^2,0,0,
0, x,0,
0, y,0,
0, z,0,
0, 0,1

a set of 7 generators given by the rows of the above matrix
\end{verbatim}

5.6.5  RepresentationMapOfRingElement

\textbf{RepresentationMapOfRingElement}(r, M, d)

\textbf{Returns}: a homalg matrix

The graded map induced by the homogeneous degree 1 ring element \( r \) (of the underlying homalg
graded ring $S$) regarded as a $R$-linear map between the $d$-th and the $(d+1)$-st homogeneous part of the graded finitely generated \texttt{homalg} $S$-module $M$, where $R$ is the coefficients ring of the graded ring $S$ with $S_0 = R$. The generating set of both modules is given by \texttt{GeneratorsOfHomogeneousPart (5.6.3)}. The entries of the matrix presenting the map lie in the coefficients ring $R$.

\begin{verbatim}
      gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
      gap> S := GradedRing( R );;
      gap> x := Indeterminate( S, 1 );
      x
      gap> M := HomalgMatrix( "[ x^3, y^2, z, z, 0, 0 ]", 2, 3, S );;
      gap> M := LeftPresentationWithDegrees( M, [ -1, 0, 1 ] );
      <A graded non-torsion left module presented by 2 relations for 3 generators>
      gap> m := RepresentationMapOfRingElement( x, M, 0 );
      <A "homomorphism" of graded left modules>
      gap> Display( m );
      1,0,0,0,0,0,0,
      0,1,0,0,0,0,0,
      0,0,0,1,0,0,0
      the graded map is currently represented by the above 3 x 7 matrix
      (degrees of generators of target: [ 1, 1, 1, 1, 1, 1, 1 ])
\end{verbatim}

\subsection{5.6.6 RepresentationMatrixOfKoszulId}

\begin{verbatim}
apiai
\end{verbatim}

\subsection{5.6.7 RepresentationMapOfKoszulId}

\begin{verbatim}
apiai
\end{verbatim}
It is assumed that all indeterminates of the underlying homalg graded ring $S$ are of degree 1. The output is the the multiplication map $\text{Hom}(A,M_d) \to \text{Hom}(A,M_{d+1})$, where $A$ is the Koszul dual ring of $S$, defined using the operation KoszulDualRing.

```
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> S := GradedRing( R );;
gap> A := KoszulDualRing( S, "a,b,c" );;
gap> M := HomalgMatrix( "[ x^3, y^2, z, z, 0, 0 ]", 2, 3, S );;
gap> M := LeftPresentationWithDegrees( M, [ -1, 0, 1 ] );
< A graded non-torsion left module presented by 2 relations for 3 generators>
gap> m := RepresentationMapOfKoszulId( 0, M );
< A homomorphism of graded left modules>
gap> Display( m );
a,b,0,0,0,0,0,0,a,b,0,0,0,0,0,0,0,a,b,c,0
the graded map is currently represented by the above 3 x 7 matrix
(degrees of generators of target: [ 4, 4, 4, 4, 4, 4, 4 ])
```

5.6.8 KoszulRightAdjoint

Returns: a homalg cocomplex

It is assumed that all indeterminates of the underlying homalg graded ring $S$ are of degree 1. Compute the homalg $A$-cocomplex $C$ of Koszul maps of the homalg $S$-module $M$ (→ RepresentationMapOfKoszulId (5.6.7)) in the $\left[ \text{degree}\text{.}\text{lowest} .. \text{degree}\text{.}\text{highest} \right]$. The Castelnuovo-Mumford regularity of $M$ is characterized as the highest degree $d$, such that $C$ is not exact at $d$. $A$ is the Koszul dual ring of $S$, defined using the operation KoszulDualRing.

```
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> S := GradedRing( R );;
gap> A := KoszulDualRing( S, "a,b,c" );;
gap> M := HomalgMatrix( "[ x^3, y^2, z, z, 0, 0 ]", 2, 3, S );;
gap> M := LeftPresentationWithDegrees( M, [ -1, 0, 1 ], S );
< A graded non-torsion left module presented by 2 relations for 3 generators>
gap> CastelnuovoMumfordRegularity( M );
1
gap> R := KoszulRightAdjoint( M, -5, 5 );
< A cocomplex containing 10 morphisms of graded left modules at degrees [-5 .. 5 ]>
gap> R := KoszulRightAdjoint( M, 1, 5 );
< An acyclic cocomplex containing 4 morphisms of graded left modules at degrees [ 1 .. 5 ]>
gap> R := KoszulRightAdjoint( M, 0, 5 );
< A cocomplex containing 5 morphisms of graded left modules at degrees [ 0 .. 5 ]>
gap> R := KoszulRightAdjoint( M, -5, 5 );
< A cocomplex containing 10 morphisms of graded left modules at degrees [-5 .. 5 ]>
```
GradedModules

GradedModules

<table>
<thead>
<tr>
<th>gap &gt; H := Cohomology( R );</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;A graded cohomology object consisting of 11 graded left modules at degrees [-5 .. 5]&gt;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>gap &gt; ByASmallerPresentation( H );</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;A non-zero graded cohomology object consisting of 11 graded left modules at degrees [-5 .. 5]&gt;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>gap &gt; Cohomology( R, -2 );</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;A graded zero left module&gt;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>gap &gt; Cohomology( R, -3 );</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;A graded zero left module&gt;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>gap &gt; Cohomology( R, -1 );</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;A graded cyclic torsion-free non-free left module presented by 2 relations for a cyclic generator&gt;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>gap &gt; Cohomology( R, 0 );</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;A graded non-zero cyclic left module presented by 3 relations for a cyclic generator&gt;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>gap &gt; Cohomology( R, 1 );</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;A graded non-zero cyclic left module presented by 2 relations for a cyclic generator&gt;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>gap &gt; Cohomology( R, 2 );</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;A graded zero left module&gt;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>gap &gt; Cohomology( R, 3 );</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;A graded zero left module&gt;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>gap &gt; Cohomology( R, 4 );</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;A graded zero left module&gt;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>gap &gt; Display( Cohomology( R, -1 ) );</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q{a,b,c}/&lt; b, a &gt; (graded, degree of generator: 0)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>gap &gt; Display( Cohomology( R, 0 ) );</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q{a,b,c}/&lt; c, b, a &gt; (graded, degree of generator: 0)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>gap &gt; Display( Cohomology( R, 1 ) );</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q{a,b,c}/&lt; b, a &gt; (graded, degree of generator: 2)</td>
</tr>
</tbody>
</table>

5.6.9 HomogeneousPartOverCoefficientsRing

<table>
<thead>
<tr>
<th>HomogeneousPartOverCoefficientsRing(d, M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Returns: a homalg module</td>
</tr>
<tr>
<td>The degree $d$ homogeneous part of the graded $R$-module $M$ as a module over the coefficient ring or field of $R$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>gap &gt; R := HomalgFieldOfRationalsInDefaultCAS( ) * &quot;x,y,z&quot;;;</td>
</tr>
<tr>
<td>gap &gt; S := GradedRing( R );;;</td>
</tr>
<tr>
<td>gap &gt; M := HomalgMatrix( &quot;[ x, y^2, z^3 ]&quot;, 3, 1, S );;;</td>
</tr>
<tr>
<td>gap &gt; M := Subobject( M, ( 1 * S )^0 );</td>
</tr>
<tr>
<td>&lt;A graded torsion-free (left) ideal given by 3 generators&gt;</td>
</tr>
<tr>
<td>gap &gt; CastelnuovoMumfordRegularity( M );</td>
</tr>
</tbody>
</table>
Graded Modules

gap> M1 := HomogeneousPartOverCoefficientsRing( 1, M );
<A graded left vector space of dimension 1 on a free generator>
gap> gen1 := GeneratorsOfModule( M1 );
<A set consisting of a single generator of a homalg left module>
gap> Display( M1 );
Q^1 (graded, degree of generator: 1)

gap> M2 := HomogeneousPartOverCoefficientsRing( 2, M );
<A graded left vector space of dimension 4 on free generators>
gap> Display( M2 );
Q^4 (graded, degrees of generators: [ 2, 2, 2, 2 ])

gap> M3 := HomogeneousPartOverCoefficientsRing( 3, M );
<A graded left vector space of dimension 9 on free generators>
gap> Display( M3 );
Q^9 (graded, degrees of generators: [ 3, 3, 3, 3, 3, 3, 3, 3, 3 ])

gap> gen3 := GeneratorsOfModule( M3 );
A set of 9 generators of a homalg left module

gap> Display( gen1 );
x

A set consisting of a single generator given by (the row of) the above matrix

gap> Display( gen2 );
x^2,
x*y,
x*z,
y^2

A set of 4 generators given by the rows of the above matrix


gap> Display( gen3 );
x^3,
x^2*y,
x^2*z,
x*y*z,
x*z^2,
x*y^2,
y^3,
y^2*z,
z^3

A set of 9 generators given by the rows of the above matrix
Chapter 6

The Tate Resolution

6.1 The Tate Resolution: Operations and Functions

6.1.1 TateResolution

\begin{verbatim}
> TateResolution(M, degree_lowest, degree_highest) (operation)

Returns: a homalg cocomplex

Compute the Tate resolution of the sheaf M.
\end{verbatim}

Example

\begin{verbatim}
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x0..x3";;
gap> S := GradedRing( R );;
gap> A := KoszulDualRing( S, "e0..e3" );;

In the following we construct the different exterior powers of the cotangent bundle shifted by 1. Observe how a single 1 travels along the diagonal in the window \([-3..0] \times [0..3]\).

First we start with the structure sheaf with its Tate resolution:
\end{verbatim}

Example

\begin{verbatim}
> 0 := S^0;
> T := TateResolution( O, -5, 5 );
> betti := BettiTable( T );
> Display( betti );

total: 35 20 10 4 1 1 4 10 20 35 56 ? ? ?
----------|---|---|---|---|---|---|---|---|---|---|---|---|---|
3: 35 20 10 4 1 . . . . . . 0 0 0
2: * . . . . . . . . . . . 0 0
1: * * . . . . . . . . . . . 0
0: * * * . . . . . 1 4 10 20 35 56
----------|---|---|---|---|---|---|---|---|---|---|---|---|---|
twist: -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5
---------------------------------------------------------------
Euler: -35 -20 -10 -4 -1 0 0 0 1 4 10 20 35 56
\end{verbatim}
The Castelnuovo-Mumford regularity of the underlying module is distinguished among the list of twists by the character 'V' pointing to it. It is not an invariant of the sheaf (see the next diagram).

The residue class field (i.e. \( S \) modulo the maximal homogeneous ideal):

```
Example

```gap
k := HomalgMatrix( Indeterminates( S ), Length( Indeterminates( S ) ), 1, S );
<A 4 x 1 matrix over a graded ring>

k := LeftPresentationWithDegrees( k );
<A graded cyclic left module presented by 4 relations for a cyclic generator>
```

Another way of constructing the structure sheaf:

```
Example

```gap
U0 := SyzygiesObject( 1, k );
<A graded torsion-free left module presented by yet unknown relations for 4 generators>

T0 := TateResolution( U0, -5, 5 );
An acyclic cocomplex containing 10 morphisms of graded left modules at degrees [-5 .. 5]>

betti0 := BettiTable( T0 );
<A Betti diagram of <An acyclic cocomplex containing 10 morphisms of graded left modules at degrees [-5 .. 5]>>

Display( betti0 );
total: 35 20 10 4 1 1 4 10 20 35 56 ? ? ?
----------|---|---|---|---|---|---|---|---|---|---|---|---|---|
3: 35 20 10 4 1 . . . . . 0 0 0
2: * . . . . . . . . . . . 0 0
1: * * . . . . . . . . . . 0 0
0: * * * . . . . . . . . . 0 0
----------|---|---|---|---|---|---|---|---|---|---|---|---|---|
twist: -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5
---------------------------------------------------------------
Euler: -35 -20 -10 -4 -1 0 0 0 1 4 10 20 35 56
```

The cotangent bundle:

```
Example

```gap
cotangent := SyzygiesObject( 2, k );
<A graded torsion-free left module presented by yet unknown relations for 6 generators>

IsFree( UnderlyingModule( cotangent ) );
false
cotangent;
<A graded reflexive non-projective rank 3 left module presented by 4 relations for 6 generators>

ProjectiveDimension( UnderlyingModule( cotangent ) );
3
cotangent;
<A graded reflexive non-projective rank 3 left module presented by 4 relations for 6 generators>
```

the cotangent bundle shifted by 1 with its Tate resolution:

```
Example

```gap
U1 := cotangent * S^1;
<A graded non-torsion left module presented by 4 relations for 6 generators>
```
GradedModules

```
gap> T1 := TateResolution( U1, -5, 5 );
<An acyclic cocomplex containing
10 morphisms of graded left modules at degrees [-5 .. 5 ]>
gap> betti1 := BettiTable( T1 );
<A Betti diagram of <An acyclic cocomplex containing
10 morphisms of graded left modules at degrees [-5 .. 5 ]>>
gap> Display( betti1 );
total: 120 70 36 15 4 1 6 20 45 84 140 ? ? ?
-----------|----|----|----|----|----|----|----|----|----|----|----|----|----|
3: 120 70 36 15 4 . . . . . 0 0 0
2: * . . . . . . . . . . . 0 0
1: * * . . . . . 1 . . . . . 0
0: * * * . . . . . . 6 20 45 84 140
-----------|----|----|----|----|----|----|----|----|----|----|----|----|----|
twist: -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5
-----------------------------------------------------------------------------
Euler: -120 -70 -36 -15 -4 0 0 -1 0 6 20 45 84 140
```

The second power $U^2$ of the shifted cotangent bundle $U = U^1$ and its Tate resolution:

```
gap> U2 := SyzygiesObject( 3, k ) * S^2;
<A graded rank 3 left module presented by 1 relation for 4 generators>
gap> T2 := TateResolution( U2, -5, 5 );
<An acyclic cocomplex containing
10 morphisms of graded left modules at degrees [-5 .. 5 ]>
gap> betti2 := BettiTable( T2 );
<A Betti diagram of <An acyclic cocomplex containing
10 morphisms of graded left modules at degrees [-5 .. 5 ]>>
gap> Display( betti2 );
total: 140 84 45 20 6 1 4 15 36 70 120 ? ? ?
-----------|----|----|----|----|----|----|----|----|----|----|----|----|----|
3: 140 84 45 20 6 . . . . . 0 0 0
2: * . . . . . 1 . . . . . 0
1: * * . . . . . . . . . . . 0
0: * * * . . . . . . 4 15 36 70 120
-----------|----|----|----|----|----|----|----|----|----|----|----|----|----|
twist: -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5
-----------------------------------------------------------------------------
Euler: -140 -84 -45 -20 -6 0 0 1 0 4 15 36 70 120
```

The third power $U^3$ of the shifted cotangent bundle $U = U^1$ and its Tate resolution:

```
gap> U3 := SyzygiesObject( 4, k ) * S^3;
<A graded free left module of rank 1 on a free generator>
gap> Display( U3 );
Q[x0,x1,x2,x3]^1
(graded, degree of generator: 1)
gap> T3 := TateResolution( U3, -5, 5 );
<An acyclic cocomplex containing
10 morphisms of graded left modules at degrees [-5 .. 5 ]>
gap> betti3 := BettiTable( T3 );
```
Another way to construct $U^2 = U^{(3-1)}$:

```gap
Example

```
Chapter 7

Examples

7.1 Betti Diagrams

7.1.1 DE-2.2  

Example

\[
gap> R := \text{HomalgFieldOfRationalsInDefaultCAS}() * "x0,x1,x2";;
\]
\[
gap> S := \text{GradedRing}( R );;
\]
\[
\text{<A 1 x 3 matrix over a graded ring>}
\]
\[
gap> \text{mat} := \text{HomalgMatrix}( \"[ x0^2, x1^2, x2^2 ]\", 1, 3, S );
\]
\[
\text{<A graded cyclic right module on a cyclic generator satisfying 3 relations>}
\]
\[
gap> M := \text{RightPresentationWithDegrees}( \text{mat}, S );
\]
\[
\text{<A graded cyclic right module on a cyclic generator satisfying 3 relations>}
\]
\[
gap> d := \text{Resolution}( M );
\]
\[
\text{<A right acyclic complex containing 3 morphisms of graded right modules at degrees [ 0 .. 3 ]>}
\]
\[
gap> \text{betti} := \text{BettiTable}( d );
\]
\[
\text{<A Betti diagram of <A right acyclic complex containing 3 morphisms of graded right modules at degrees [ 0 .. 3 ]>}> 
\]
\[
gap> \text{Display}( \text{betti} );
\]
\[
\text{total: 1 3 3 1}
\]
\[
\begin{array}{l}
0: 1 . . . \\
1: . 3 . . \\
2: . . 3 . \\
3: . . . 1
\end{array}
\]
\[
\text{degree: 0 1 2 3}
\]
\[
gap> ## we are still below the Castelnuovo-Mumford regularity, which is 3:
\]
\[
gap> M2 := \text{SubmoduleGeneratedByHomogeneousPart}( 2, M );
\]
\[
\text{<A graded torsion right submodule given by 3 generators>}
\]
\[
gap> d2 := \text{Resolution}( M2 );
\]
\[
\text{<A right acyclic complex containing 3 morphisms of graded right modules at degrees [ 0 .. 3 ]>}
\]
\[
gap> \text{betti2} := \text{BettiTable}( d2 );
\]
\[
\text{<A Betti diagram of <A right acyclic complex containing 3 morphisms of graded right modules at degrees [ 0 .. 3 ]>}> 
\]
\[
gap> \text{Display}( \text{betti2} );
\]
\[
\text{total: 3 8 6 1}
\]
7.1.2 DE-Code

Example

```gap
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x0,x1,x2";;
gap> S := GradedRing( R );;
gap> mat := HomalgMatrix( "[ x0^2, x1^2 ]", 1, 2, S );;
<A 1 x 2 matrix over a graded ring>
gap> M := RightPresentationWithDegrees( mat, S );;
<A graded cyclic right module on a cyclic generator satisfying 2 relations>
gap> d := Resolution( M );;
<A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>
gap> betti := BettiTable( d );
<A Betti diagram of <A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>>
gap> Display( betti );
total: 1 2 1
--------------
0: 1 . .
1: . 2 .
2: . . 1
--------------
degree: 0 1 2
```

7.1.3 Schenck-3.2

This is an example from Section 3.2 in [Sch03].

```gap
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> mmat := HomalgMatrix( "[ x, x^3 + y^3 + z^3 ]", 1, 2, Qxyz );
<A 1 x 2 matrix over an external ring>
gap> S := GradedRing( Qxyz );
gap> M := RightPresentationWithDegrees( mmat, S );
<A graded cyclic right module on a cyclic generator satisfying 2 relations>
```
```gap
Mr := Resolution( M );
<A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>

bettiM := BettiTable( Mr );
<A Betti diagram of <A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>>

Display( bettiM );
total: 1 2 1
-------------
0: 1 1 .
1: . . .
2: . 1 1
-------------
degree: 0 1 2

R := GradedRing( CoefficientsRing( S ) * "x,y,z,w" );;

nmat := HomalgMatrix( "[ z^2 - y*w, y*z - x*w, y^2 - x*z ]", 1, 3, R );
<A 1 x 3 matrix over a graded ring>

N := RightPresentationWithDegrees( nmat );
<A graded cyclic right module on a cyclic generator satisfying 3 relations>

Nr := Resolution( N );
<A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>

bettiN := BettiTable( Nr );
<A Betti diagram of <A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>>

Display( bettiN );
total: 1 3 2
-------------
0: 1 . .
1: . 3 2
-------------
degree: 0 1 2
```

### 7.1.4 Schenck-8.3

This is an example from Section 8.3 in [Sch03].

```gap
R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z,w";;

S := GradedRing( R );;

jmat := HomalgMatrix( "[ z*w, x*w, y*z, x*y, x^3*z - x*z^3 ]", 1, 5, S );
<A 1 x 5 matrix over a graded ring>

J := RightPresentationWithDegrees( jmat );
<A graded cyclic right module on a cyclic generator satisfying 5 relations>

Jr := Resolution( J );
<A right acyclic complex containing
3 morphisms of graded right modules at degrees [ 0 .. 3 ]>

betti := BettiTable( Jr );
<A Betti diagram of <A right acyclic complex containing
3 morphisms of graded right modules at degrees [ 0 .. 3 ]>>

Display( betti );
total: 1 5 6 2
-------------
```
7.1.5 Schenck-8.3.3

This is Exercise 8.3.3 in [Sch03].

Example

```gap
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> S := GradedRing( Qxyz );;
gap> mat := HomalgMatrix( "[ x*y*z, x*y^2, x^-2*z, x^-2*y, x^-3 ]", 1, 5, S );;
<A 1 x 5 matrix over a graded ring>
gap> M := RightPresentationWithDegrees( mat, S );;
<A right cyclic right module on a cyclic generator satisfying 5 relations>
gap> Mr := Resolution( M );;
<A right acyclic complex containing
3 morphisms of graded right modules at degrees [ 0 .. 3 ]>
gap> betti := BettiTable( Mr );
<A Betti diagram of <A right acyclic complex containing
3 morphisms of graded right modules at degrees [ 0 .. 3 ]>>
gap> Display( betti );
  total:  1  5  6  2
  degree: 0  1  2  3
```

7.2 Commutative Algebra

7.2.1 Saturate

Example

```gap
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> S := GradedRing( R );;
gap> m := GradedLeftSubmodule( "x,y,z", S );
<A graded torsion-free (left) ideal given by 3 generators>
gap> I := Intersect( m^3, GradedLeftSubmodule( "x", S ) );
<A graded torsion-free (left) ideal given by 6 generators>
gap> NrRelations( I );
  8
  gap> Im := SubobjectQuotient( I, m );
<A graded torsion-free rank 1 (left) ideal given by 3 generators>
gap> I_m := Saturate( I, m );
<A graded principal (left) ideal of rank 1 on a free generator>
gap> Is := Saturate( I );
<A graded principal (left) ideal of rank 1 on a free generator>
gap> Assert( 0, Is = I_m );
```
7.3 Global Section Modules of the Induced Sheaves

7.3.1 Examples of the ModuleOfGlobalSections Functor and Purity Filtrations

Example

```gap
gap> LoadPackage( "GradedRingForHomalg" );;
gap> Qxyzt := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z,t";;
gap> S := GradedRing( Qxyzt );;
gap>
gap> wmat := HomalgMatrix( 
"[ 
> x*y, y*z, z*t, 0, 0, 0,
> x^3*z, x^2*z^2, 0, x*z^2*t, 0, 0,
> x^4, x^3*z, 0, x^2*z^2, 0, 0,
> 0, 0, x*y, -y^2, x^2-t^2, 0,
> 0, 0, x^2*z, -x*y*z, y*z*t, 0,
> 0, 0, x^2*y-x^2*t,-x*y^2+x*y*t, y^2*t-y*t^2, 0,
> 0, 0, 0, 0, -1, 1 
]", 7, 6, Qxyzt );;
gap>
gap> LoadPackage( "GradedModules" );;
gap> wmor := GradedMap( wmat, "free", "free", "left", S );;
gap> IsMorphism( wmor );;
gap> W := LeftPresentationWithDegrees( wmat, S );;
gap> HW := ModuleOfGlobalSections( W );
<A graded left module presented by yet unknown relations for 6 generators>
gap> LinearStrandOfTateResolution( W, 0,4 );
<A cocomplex containing 4 morphisms of graded left modules at degrees [ 0 .. 4 ]>
gap> purity_iso := IsomorphismOfFiltration( PurityFiltration( W ) );
<A non-zero isomorphism of graded left modules>
gap> Hpurity_iso := ModuleOfGlobalSections( purity_iso );
<An isomorphism of graded left modules>
gap> ModuleOfGlobalSections( wmor );
<An isomorphism of graded left modules>
```

7.3.2 Horrocks Mumford bundle

This example computes the global sections module of the Horrocks-Mumford bundle.

Example

```gap
gap> LoadPackage( "GradedRingForHomalg" );;
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x0..x4";;
gap> S := GradedRing( R );;
gap> A := KoszulDualRing( S, "e0..e4" );;
gap> LoadPackage( "GradedModules" );;
gap> mat := HomalgMatrix( 
"[ 
> e1*e4, e2*e0, e3*e1, e4*e2, e0*e3, 
> e2*e3, e3*e4, e4*e0, e0*e1, e1*e2 
]",
> 2, 5, A );
<A 2 x 5 matrix over a graded ring>
gap> phi := GradedMap( mat, "free", "free", "left", A );;
```
gap> IsMorphism( phi );
true

# I GuessModuleOfGlobalSectionsFromATateMap uses a heuristic for efficiency;
please check the correctness of the following result

gap> IsPure( M );
true

gap> Rank( M );
2

gap> Display( BettiTable( Resolution( M ) ) );
total:  19  35  20  2
---------------------
3:    4 . . .
4:  15  35  20 .
5: . . . 2
---------------------
degree: 0  1  2  3

gap> Display( BettiTable( TateResolution( M, -5, 5 ) ) );
total: 100  37  14  10  5  2  5  10  14  37  100  ?  ?  ? ?
---------------------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
4:  100  35  4 . . . . . . 0  0  0  0
3:  *  2 10 10  5 . . . . . 0  0  0
2:  * * . . . . 2 . . . . 0  0
1:  * * * . . . . 5 10 10  2 . 0
0:  * * * * . . . . . . 4 35 100
---------------------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
twist:  -9  -8  -7  -6  -5  -4  -3  -2  -1  0  1  2  3  4  5

Euler: 100  35  2 -10 -10 -5  0  2  0 -5 -10 -10  2  35  100

gap> M;

<A graded reflexive non-projective rank 2 left module presented by 94 \nrelations for 19 generators>

gap> P := ElementOfGrothendieckGroup( M );
( 2*O_{P^4} - 1*O_{P^3} - 4*O_{P^2} - 2*O_{P^1} ) -> P^4

gap> P!.DisplayTwistedCoefficients := true;
true

gap> P;
( 2*O(-3) - 10*O(-2) + 15*O(-1) - 5*O(0) ) -> P^4

gap> chi := HilbertPolynomial( M );
1/12*t^4+2/3*t^3-1/12*t^2-17/3*t-5

gap> c := ChernPolynomial( M );
( 2 | 1-h+4*h^2 ) -> P^4

[ 2-u-7*u^2/2!+11*u^3/3!+17*u^4/4! ] -> P^4

gap> ch := ChernCharacter( M );
[ -8 .. 7 ], i -> Value( chi, i ) ;

[ 35, 2, -10, -10, -5, 0, 2, 0, -5, -10, -10, 2, 35, 100, 210, 380 ]
function( t ) ... end

gap> List( [ 0 .. 7 ], HF );
[ 0, 0, 0, 4, 35, 100, 210, 380 ]

gap> IndexOfRegularity( M );
4

gap> DataOfHilbertFunction( M );
[ [ [ 4 ], [ 3 ] ], 1/12*t^4+2/3*t^3-1/12*t^2-17/3*t-5 ]
Appendix A

Overview of the GradedModules Package Source Code
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