GAP 4 Package Thelma

THreshold ELements, Modeling and Applications

1.0.2

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Victor Bovdi

Vasyl Laver

Victor Bovdi
Email: vbovdi@gmail.com
Address: Department of Mathematical Sciences
UAEU
Al Ain, United Arab Emirates

Vasyl Laver
Email: vasyl.laver@uzhnu.edu.ua
Address: Department of Mathematical Sciences
UAEU
Al Ain, United Arab Emirates

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Chapter 1

Introduction

Thelma stands for “THreshold ELements, Modelling and Applications". This package is dedicated to realization of Boolean functions by the means of the threshold elements and multilayered perceptrons. Threshold elements were introduced in [MP43]. A certain number of articles on threshold logic appeared in 60-s and 70-s, however this field is regaining a considerable interest nowadays as well (see [HST16],[Hor94], [GVKB11], [KYSV16], e.t.c).

We mostly refer to the methods proposed in [ABGG80], [GPR83], [GMB17], and [Der65].

1.1 Overview over this manual

Chapter 3 describes the functions operating with the threshold elements. They include the basic operations, the functions that verify the realizability of a given Boolean function by a single threshold element, and some iterative methods. Chapter 4 describes the functions for neural networks, built from the threshold elements.

1.2 Installation

To get the newest version of this GAP 4 package download the archive file `thelma-x.x.tar.gz` and unpack it in a directory called “`pkg`”, preferably (but not necessarily) in the “`pkg`” subdirectory of your GAP 4 installation. It creates a subdirectory called “`thelma`”.

As Thelma has no additional C libraries, there is no need in any additional installation steps.

1.3 Feedback

For bug reports, feature requests and suggestions, please use the github issue tracker at `https://github.com/gap-packages/Thelma/issues`.

Chapter 2

Boolean Functions

2.1 Basic Operations

Let $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ be a Boolean function. The vector

$$F = (\, f(0),\, f(1),\, \ldots,\, f(2^n - 1)\,)^T,$$

where $f(i)$ for each $i \in \{0, 1, \ldots, 2^n - 1\}$ is the value of $f(x_1, \ldots, x_n)$ of the i-th row in the truth table, is called the *truth vector*.

As a generalization of Boolean logic we can consider the k-valued logic, thus $f : \mathbb{Z}_k^n \to \mathbb{Z}_k$. Other way to generalize the concept of Boolean functions is the introduction of discrete logic functions, defined in Chapter 5.

2.1.1 LogicFunction

▷ `LogicFunction(`*`NumVars, Dimension, Output`*`)` (function)

For the positive integer `NumVars` - the number of variables, a positive integer `Dimension` - the number of possible logical values and a list non-negative integers `Output` - the truth vector of the given `Dimension`-valued logic function of `NumVars` variables, the function `LogicFunction` returns an object, representing the corresponding logic function.

Note that `Dimension` can be also a list of `NumVars` positive integer numbers if we deal with discrete logic functions.

Example

```
gap> f:=LogicFunction(2,2,[0,0,1,1]);
< Boolean function of 2 variables >
gap> Display(f);
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 0
[ 1, 0 ] || 1
[ 1, 1 ] || 1
gap> f:=LogicFunction(2,3,[0,0,1,1,2,1,2,0,1]);
< 3-valued logic function of 2 variables >
gap> Display(f);
3-valued logic function of 2 variables.
```

```
[ 0, 0 ] || 0
[ 0, 1 ] || 0
[ 0, 2 ] || 1
[ 1, 0 ] || 1
[ 1, 1 ] || 2
[ 1, 2 ] || 1
[ 2, 0 ] || 2
[ 2, 1 ] || 0
[ 2, 2 ] || 1
```

2.1.2 IsLogicFunction

▷ `IsLogicFunction(`*`Obj`*`)` (function)

For the object `Obj` the function `IsLogicFunction` returns `true` if `Obj` is a logic function (see `LogicFunction` (2.1.1)), and `false` otherwise.

Example

```
gap> f:=LogicFunction(2,2,[0,1,1,1]);;
gap> IsLogicFunction(f);
true
```

2.1.3 PolynomialToBooleanFunction

▷ `PolynomialToBooleanFunction(`*`Pol, NumVars`*`)` (function)

For the polynomial `Pol` over $GF(2)$ and the number of variables `NumVar` the function `PolynomialToBooleanFunction` returns a Boolean logic function which is realized by `Pol`.

Example

```
gap> x:=Indeterminate(GF(2),"x");;
gap> y:=Indeterminate(GF(2),"y");;
gap> pol:=x+y;;
gap> f:=PolynomialToBooleanFunction(pol,2);
< Boolean function of 2 variables >
gap> Display(f);
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 1
[ 1, 0 ] || 1
[ 1, 1 ] || 0
```

2.1.4 IsUnateInVariable

▷ `IsUnateInVariable(`*`Func, Var`*`)` (function)

A Boolean function $f(x_1,\ldots,x_n)$ is *positive unate* in x_i if for all possible values of x_j with $j \neq i$ we have

$$f(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n) \geq f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n).$$

A Boolean function $f(x_1,\ldots,x_n)$ is *negative unate* in x_i if

$$f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n) \geq f(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n).$$

For the Boolean function `Func` and the positive integer `Var` (which represents the number of the variable) the function `IsUnateBooleanFunction` returns `true` if `Func` is unate (either positive or negative) in this variable and `false` otherwise.

Example

```
gap> f:=LogicFunction(3,2,[0,0,0,0,0,1,1,0]);
< Boolean function of 3 variables >
gap> Display(f);
Boolean function of 3 variables.
[ 0, 0, 0 ] || 0
[ 0, 0, 1 ] || 0
[ 0, 1, 0 ] || 0
[ 0, 1, 1 ] || 0
[ 1, 0, 0 ] || 0
[ 1, 0, 1 ] || 1
[ 1, 1, 0 ] || 1
[ 1, 1, 1 ] || 0
gap> IsUnateInVariable(f,1);
true
gap> IsUnateInVariable(f,2);
false
gap> IsUnateInVariable(f,3);
false
```

2.1.5 IsUnateBooleanFunction

▷ `IsUnateBooleanFunction(`*Func*`)` (function)

If a Boolean function f is either positive or negative unate in each variable then it is said to be *unate* (note that some x_i may be positive unate and some negative unate to satisfy the definition of unate function). A Boolean function f is *binate* if it is not unate (i.e., is neither positive unate nor negative unate in at least one of its variables).

All threshold functions are unate. However, the converse is not true, because there are certain unate functions, that can not be realized by STE [AQR99].

For the Boolean function `Func` the function `IsUnateBooleanFunction` returns `true` if `Func` is unate and `false` otherwise.

Example

```
gap> f:=LogicFunction(2,2,[0,1,1,1]);;
gap> IsUnateBooleanFunction(f);
true
gap> f:=LogicFunction(2,2,[0,1,1,0]);;
```

```
gap> IsUnateBooleanFunction(f);
false
```

2.1.6 InfluenceOfVariable

▷ `InfluenceOfVariable(`*`Func, Var`*`)` (function)

The influence of a variable x_i measures how many times out of the total existing cases a change on that variable produces a change on the output of the function.

For the Boolean function `Func` and the positive integer `Var` the function `InfluenceOfVariable` returns a positive integer - the weighted influence of the variable `Var` (to obtain integer values we multiply the influence of the variable by 2^n, where n is the number of variables of `Func`).

Example

```
gap> f:=LogicFunction(3,2,[0,0,0,0,0,1,1,0]);
< Boolean function of 3 variables >
gap> Display(f);
Boolean function of 3 variables.
[ 0, 0, 0 ] || 0
[ 0, 0, 1 ] || 0
[ 0, 1, 0 ] || 0
[ 0, 1, 1 ] || 0
[ 1, 0, 0 ] || 0
[ 1, 0, 1 ] || 1
[ 1, 1, 0 ] || 1
[ 1, 1, 1 ] || 0
gap> InfluenceOfVariable(f,2);
2
```

2.1.7 SelfDualExtensionOfBooleanFunction

▷ `SelfDualExtensionOfBooleanFunction(`*`Func`*`)` (function)

The *`self-dual extension`* of a Boolean function $f^n : \mathbb{Z}_2^n \to \mathbb{Z}_2$ of n variables is a Boolean function $f^{n+1} : \mathbb{Z}_2^{n+1} \to \mathbb{Z}_2$ of $n+1$ variables defined as

$$f^{n+1}(x_1,\ldots,x_n,x_{n+1}) = f^n(x_1,\ldots,x_n) \quad \text{if} \quad x_{n+1} = 0,$$

$$f^{n+1}(x_1,\ldots,x_n,x_{n+1}) = 1 - f^n(\overline{x}_1,\ldots,\overline{x}_n) \quad \text{if} \quad x_{n+1} = 1,$$

where $\overline{x}_i = x_i \oplus 1$ is the negation of the i-th variable.

Every threshold function is unate. However, in [FSAJ06] was shown that the unatness in the self-dual space of $n+1$ variables is much stronger condition.

For the Boolean function `Func` the function `SelfDualExtensionOfBooleanFunction` returns the self-dual extension of `Func`.

Example

```
gap> f:=LogicFunction(2,2,[0,0,0,1]);
< Boolean function of 2 variables >
gap> Display(f);
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 0
[ 1, 0 ] || 0
[ 1, 1 ] || 1
gap> fsd:=SelfDualExtensionOfBooleanFunction(f);
< Boolean function of 3 variables >
gap> Display(fsd);
Boolean function of 3 variables.
[ 0, 0, 0 ] || 0
[ 0, 0, 1 ] || 0
[ 0, 1, 0 ] || 0
[ 0, 1, 1 ] || 1
[ 1, 0, 0 ] || 0
[ 1, 0, 1 ] || 1
[ 1, 1, 0 ] || 1
[ 1, 1, 1 ] || 1
```

2.1.8 SplitBooleanFunction

▷ `SplitBooleanFunction(`*Func, Var, Bool*`)` (function)

The method of splitting a function in terms of a given variable is known as Shannon decomposition and it was formally introduced in 1938 by Shannon.

Let $f(x_1,\ldots,x_n)$ be a Boolean function. Decompose f as a disjunction of the following two Boolean functions f_a and f_b defined as:

$$f_a(x_1,\ldots,x_n) = f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n) \quad \text{if} \quad x_i = 0,$$

$$f_a(x_1,\ldots,x_n) = 0, \quad \text{if} \quad x_i = 1;$$

and

$$f_b(x_1,\ldots,x_n) = 0 \quad \text{if} \quad x_i = 0,$$

$$f_b(x_1,\ldots,x_n) = f(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n) \quad \text{if} \quad x_i = 1.$$

If we are intended to use conjunction, we can apply the same equations with 1 for undetermined outputs instead of 0.

For the Boolean function `Func`, a positive integer `Var` (the number of variable), Boolean variable `Bool` (`true` for disjunction and `false` for conjunction) the function `SplitBooleanFunction` returns a list with two entries: the resulting Boolean logic functions.

Example

```
gap> f:=LogicFunction(2,2,[0,1,1,0]);;
gap> Display(f);
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 1
[ 1, 0 ] || 1
[ 1, 1 ] || 0
gap> out:=SplitBooleanFunction(f,1,false);;
gap> Print(out[1]);
[2, 2,[ 0, 1, 1, 1 ]]
gap> Print(out[2]);
[2, 2,[ 1, 1, 1, 0 ]]
gap> out:=SplitBooleanFunction(f,1,true);;
gap> Print(out[1]);
[2, 2,[ 0, 1, 0, 0 ]]
gap> Print(out[2]);
[2, 2,[ 0, 0, 1, 0 ]]
```

2.1.9 KernelOfBooleanFunction

▷ `KernelOfBooleanFunction(`*Func*`)` (function)

For a Boolean function $f(x_1,\ldots,x_n)$ we define the following two sets (see [ABGG80]):

$$f^{-1}(1) = \{\, \mathbf{x} \in \mathbb{Z}_2^n \mid f(\mathbf{x}) = 1 \,\}, \quad \text{and} \quad f^{-1}(0) = \{\, \mathbf{x} \in \mathbb{Z}_2^n \mid f(\mathbf{x}) = 0 \,\}.$$

The kernel $K(f)$ of the Boolean function f is defined as

$$K(f) = f^{-1}(1), \quad \text{if} \quad |f^{-1}(1)| \geq |f^{-1}(0)|;$$

$$K(f) = f^{-1}(0), \quad \text{otherwise,}$$

where $|f^{-1}(i)|$ is the cardinality of the set $f^{-1}(i)$ with $i \in \{0,1\}$.

For the Boolean function `Func` the function `KernelOfBooleanFunction` returns a list in which the first element of the output list represents the kernel, and the second element equals either 1 or 0.

Example

```
gap> f:=LogicFunction(3,2,[0,1,1,0,1,0,0,0]);;
gap> k:=KernelOfBooleanFunction(f);
[ [ [ 0, 0, 1 ], [ 0, 1, 0 ], [ 1, 0, 0 ] ], 1 ]
```

2.1.10 ReducedKernelOfBooleanFunction

▷ `ReducedKernelOfBooleanFunction(`*Ker*`)` (function)

Let $f(x_1,\ldots,x_n)$ be a Boolean function with the kernel $K(f) = \{ a_1,\ldots,a_m \}$, where $m \leq 2^{n-1}$. The reduced kernel $K(f)_i$ of the function f relative to the element $a_i \in K(f)$ is the following set (see [ABGG80]):

$$K(f)_i = \{ a_1 \oplus a_i,\ a_2 \oplus a_i,\ \ldots,\ a_m \oplus a_i \},$$

where $\oplus$ is a component-wise addition of vectors from $K(f)$ over $GF(2)$.

The reduced kernel $T(f)$ of f is the following set:

$$T(f) = \{ K(f)_i \mid i = 1,2,\ldots,m \}.$$

For the $m \times n$ matrix `Ker`, which represents the kernel of some Boolean function f, the function `ReducedKernelOfBooleanFunction` returns the reduced kernel $T(f)$ of f.

Example

```
gap> ## Continuation of Example 2.2.4
gap> rk:=ReducedKernelOfBooleanFunction(k[1]);;
gap> j:=1;;
gap> for i in rk do Print(j,".\n"); Display(i); Print("\n"); j:=j+1; od;
1.
 . . .
 . 1 1
 1 . 1

2.
 . 1 1
 . . .
 1 1 .

3.
 1 . 1
 1 1 .
 . . .
```

Chapter 3

Threshold Elements

3.1 Basic Operations

For a given real vector $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ and a threshold $T \in \mathbb{R}$, the *threshold element* is a function $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ defined by the following relations:

$$f(x_1, \ldots, x_n) = 1, \quad \text{if} \quad \sum_{i=1}^{n} w_i x_i \geq T, \quad \text{and} \quad f(x_1, \ldots, x_n) = 0 \quad \text{otherwise},$$

in which $f(x_1, \ldots, x_n)$ is the binary output (valued 0 or 1), each variable x_i is the i-th input (valued 0 or 1), and n is the number of inputs.

The vector w is the *weight* vector, and the $x = (x_1, \ldots, x_n)$ is the *input* vector. The vector $(w_1, \ldots, w_n; T)$ is called the *structure vector* (or simply the *structure*) of the threshold element.

3.1.1 ThresholdElement

▷ `ThresholdElement(`*Weights, Threshold*`)` (function)

For the list of rational numbers `Weights` and the rational `Threshold` the function `ThresholdElement` returns a threshold element with the number of inputs equal to the length of the `Weights` list.

Example

```
gap> te:=ThresholdElement([1,2],3);
< threshold element with weight vector [ 1, 2 ] and threshold 3 >
gap> Display(te);
Weight vector = [ 1, 2 ], Threshold = 3.
Threshold Element realizes the function f :
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 0
[ 1, 0 ] || 0
[ 1, 1 ] || 1
Sum of Products:[ 3 ]
```

The function Display outputs the stucture of the given threshold element ThrEl and the Sum of Products or Product of Sums representation of the function realized by ThrEl. For threshold elements of $n \leq 4$ variables it also prints the truth table of the realized Boolean function.

Example

```
gap> w:=[1,2,4,-4,6,8,10,-25,6,32];;
gap> T:=60;;
gap> te:=ThresholdElement(w,T);
< threshold element with weight vector [ 1, 2, 4, -4, 6, 8, 10, -25, 6, 32
 ] and threshold 60 >
gap> Display(te);
Weight vector = [ 1, 2, 4, -4, 6, 8, 10, -25, 6, 32 ], Threshold = 60.
Threshold Element realizes the function f :
Sum of Products:[ 59, 155, 185, 187, 251, 315, 379, 411, 427, 441, 443, 507, 5\
71, 667, 697, 699, 763, 827, 891, 923, 939, 953, 955, 1019 ]
```

3.1.2 IsThresholdElement

▷ IsThresholdElement(*Obj*) (function)

For the object Obj the function IsThresholdElement returns true if Obj is a threshold element (see ThresholdElement (3.1.1)), and false otherwise.

Example

```
gap> te:=ThresholdElement([1,2],3);
< threshold element with weight vector [ 1, 2 ] and threshold 3 >
gap> IsThresholdElement(te);
true
gap> IsThresholdElement([[1,2],3]);
false
```

3.1.3 OutputOfThresholdElement

▷ OutputOfThresholdElement(*ThrEl*) (function)

For the threshold element ThrEl the function OutputOfThresholdElement returns the Boolean function, realized by ThrEl.

Example

```
gap> te:=ThresholdElement([1,2],3);
< threshold element with weight vector [ 1, 2 ] and threshold 3 >
gap> f:=OutputOfThresholdElement(te);
< Boolean function of 2 variables >
gap> Display(f);
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 0
[ 1, 0 ] || 0
```

```
[ 1, 1 ] || 1
```

3.1.4 StructureOfThresholdElement

▷ StructureOfThresholdElement(*ThrEl*) (function)

For the threshold element ThrEl the function StructureOfThresholdElement returns the structure vector [Weights,Threshold] (see ThresholdElement (3.1.1)).

Example

```
gap> te:=ThresholdElement([1,2],3);
< threshold element with weight vector [ 1, 2 ] and threshold 3 >
gap> sv:=StructureOfThresholdElement(te);
[ [ 1, 2 ], 3 ]
```

3.1.5 RandomThresholdElement

▷ RandomThresholdElement(*NumVar, Lo, Hi*) (function)

For the integers NumVar, Lo, and Hi, the function RandomThresholdElement returns a threshold element of NumVar variables with a pseudo random integer weight vector and an integer threshold, where both the weights and the threshold are chosen from the interval [Lo, Hi].

Example

```
gap> te:=RandomThresholdElement(4,-10,10);
< threshold element with weight vector [ 7, -8, -6, 10 ] and threshold 2 >
```

3.1.6 Comparison of Threshold Elements

▷ Comparison of Threshold Elements(*ThrEl1, ThrEl2*) (function)

Let ThrEl1 and ThrEl2 be two threshold elements of the same number of variables, which realize the following Boolean functions (see ThresholdElement (3.1.1)) f_1 and f_2, resprectively. By comparison of two threshold elements we mean the comparison of the truth vectors of f_1 and f_2 (see OutputOfThresholdElement (3.1.3)).

Example

```
gap> te1:=ThresholdElement([1,2],3);;
gap> Print(OutputOfThresholdElement(te1),"\n");
[2, 2,[ 0, 0, 0, 1 ]]
gap> te2:=ThresholdElement([1,2],0);;
gap> Print(OutputOfThresholdElement(te2),"\n");
[2, 2,[ 1, 1, 1, 1 ]]
gap> te3:=ThresholdElement([1,1],2);;
gap> Print(OutputOfThresholdElement(te3),"\n");
[2, 2,[ 0, 0, 0, 1 ]]
```

```
gap> te1<te2;
true
gap> te1>te2;
false
gap> te1=te3;
true
```

3.2 Single Threshold Element Realizability

One of the most important questions is whether a Boolean function can be realized by a single threshold element (STE). A Boolean function which is realizable by a STE is called a `Threshold Function`. This section is dedicated to verification of STE-realizability.

3.2.1 CharacteristicVectorOfFunction

▷ `CharacteristicVectorOfFunction(`*Func*`)` (function)

Let $f(x_1,\ldots,x_n)$ be a Boolean function. We can switch from the {0,1}-base to {-1,1}-base using the following transformation:

$$y_i = 2x_i - 1, \quad (i = 1,2,\ldots,n)$$

$$g(y_1,\ldots,y_n) = 2f(x_1,\ldots,x_n) - 1.$$

For each $i \in \{1,2,\ldots,n\}$ the i-th column of the truth table of the function $g(y_1,\ldots,y_n)$ (in {-1,1}-base) we denote by Y_i, and the truth vector of g we denote by G.

Define the following vector:

$$b = \left(Y_1 \cdot G,\ \ldots,\ Y_n \cdot G,\ \textstyle\sum_{i=0}^{2^n-1} g(i) \right) \in \mathbb{R}^{n+1},$$

where $Y_k \cdot G$ is the classical inner (scalar) product for each $k \in \{1,\ldots,n\}$.

Vector b is called the *characteristic vector* of the Boolean function f [Der65]. Comparing the characteristic vector of the function f with the lists of characteristic vectors of all STE-realizable functions we obtain the answer wheter f is realizable by STE or not. In Thelma package we have a database of all such vectors for STE-realizable functions of $n \leq 6$ variables obtained from [Der65]. For the Boolean function `Func` the function `CharacteristicVectorOfFunction` returns a characteristic vector. There are no limitations on the cardinality of `Func`, but the database of STE-realizable functions is given only for $n \leq 6$ variables.

Example

```
gap> f:=LogicFunction(2,2,[0,0,0,1]);
< Boolean function of 2 variables >
gap> CharacteristicVectorOfFunction(f);
[ 2, 2, 2 ]
```

3.2.2 IsCharacteristicVectorOfSTE

▷ `IsCharacteristicVectorOfSTE(`*ChVect*`)` (function)

For the characteristic vector `ChVect` (see `CharacteristicVectorOfFunction` (3.2.1)) the function `IsCharacteristicVectorOfSTE` returns `true` if `ChVect` is a characteristic vector of some STE-realizable Boolean function, and `false` otherwise. Note, that this function is implemented only for characteristic vectors of length not bigger than 7.

Example

```
gap> f:=LogicFunction(2,2,[0,0,0,1]);
< Boolean function of 2 variables >
gap> c:=CharacteristicVectorOfFunction(f);
[ 2, 2, 2 ]
gap> IsCharacteristicVectorOfSTE(c);
true
```

3.2.3 IsInverseInKernel

▷ `IsInverseInKernel(`*Func*`)` (function)

Let $f(x_1,\ldots,x_n)$ be a Boolean function with the kernel $K(f)$. The function `IsInverseInKernel` returns `true` if there is a pair of additive inverse vectors in $K(f)$ (this means that f is not STE-realizable, see [GPR83]) or `false` otherwise. Note that this function also accepts the kernel of the Boolean function `Func` as an input. A vector $b \in \mathbb{Z}_2^n$ is called an additive inverse to $a \in \mathbb{Z}_2^n$ if $a \oplus b = 0$.

Example

```
gap> f:=LogicFunction(3,2,[0,1,0,1,0,1,1,0]);;
gap> k:=KernelOfBooleanFunction(f);;
gap> Display(k[1]);
 . . 1
 . 1 1
 1 . 1
 1 1 .
gap> IsInverseInKernel(f);
true
```

3.2.4 IsKernelContainingPrecedingVectors

▷ `IsKernelContainingPrecedingVectors(`*Func*`)` (function)

A vector $a = (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}_2^n$ precedes a vector $b = (\beta_1,\ldots,\beta_n) \in \mathbb{Z}_2^n$ (we denote it as $a \prec b$) if $\alpha_i \leq \beta_i$ for each $i = 1,\ldots,n$.

For a given vector $c \in \mathbb{Z}_2^n$ denote $M_c = \{\, a \in \mathbb{Z}_2^n \mid a \prec c \,\}$.

Let $f(x_1,\ldots,x_n)$ be a Boolean function with reduced kernel $T(f) = \{K(f)_j \mid j = 1,2,\ldots,m\}$. If f is implemented by a single threshold element (STE), then there exists $j \in \{1,\ldots,m\}$ such that

$$\forall a \in K(f)_j \qquad \text{holds} \qquad M_a \subseteq K(f)_j.$$

The function IsKernelContainingPrecedingVectors returns false for a given function Func if *Func* is not realizable by a single threshold element (see [GMB17]). Note that this function also accepts the kernel of the Boolean function Func as an input.

Example

```
gap> ##Continuation of the previous example
gap> IsKernelContainingPrecedingVectors(f);
false
```

3.2.5 IsRKernelBiggerOfCombSum

▷ IsRKernelBiggerOfCombSum(*Func*) (function)

Let $f(x_1,\ldots,x_n)$ be a Boolean function with reduced kernel $T(f)$. Denote

$$k_i^* = \max\left\{ \|a\| = \sum_{j=1}^{m} a_j \mid a = (a_1,\ldots,a_m) \in T(f) \right\}, \quad (i = 1,\ldots,n)$$

and

$$k_A^* = \min\left\{ k_i^* \mid i = 1,2,\ldots,n \right\}.$$

If f is implemented by a single threshold element (STE), then the following condition holds:

$$|A| \geq \sum_{i=0}^{k_A^*} \binom{k_A^*}{i},$$

where $\binom{k_A^*}{i}$ is the classical binomial coefficient and $|A|$ is the cardinality of A.

For a given Boolean function Func the function IsRKernelBiggerOfCombSum returns false if this function is not STE-realizable (see [GMB17]). Note that this function also accepts the reduced kernel of the Boolean function Func as an input.

Example

```
gap> f:=LogicFunction(2,2,[0,1,1,0]);
< Boolean function of 2 variables >
gap> IsRKernelBiggerOfCombSum(f);
false
```

3.2.6 BooleanFunctionBySTE

▷ BooleanFunctionBySTE(*Func*) (function)

For a given Boolean function Func the function BooleanFunctionBySTE determines whether Func is realizable by a single threshold element (STE). The function returns a threshold element with integer weights and integer threshold. If Func is not realizable by STE, it returns an empty list []. The realization of the function BooleanFunctionBySTE is based on algorithms, proposed in [Gec10].

Example

```
gap> f:=LogicFunction(3,2,[1,1,0,0,1,0,0,0]);
```

```
< Boolean function of 3 variables >
gap> te:=BooleanFunctionBySTE(f);
< threshold element with weight vector [ -1, -4, -2 ] and threshold -2 >
gap> Display(te);
Weight vector = [ -1, -4, -2 ], Threshold = -2.
Threshold Element realizes the function f :
Boolean function of 3 variables.
[ 0, 0, 0 ] || 1
[ 0, 0, 1 ] || 1
[ 0, 1, 0 ] || 0
[ 0, 1, 1 ] || 0
[ 1, 0, 0 ] || 1
[ 1, 0, 1 ] || 0
[ 1, 1, 0 ] || 0
[ 1, 1, 1 ] || 0
Sum of Products:[ 0, 1, 4 ]
gap> f:=LogicFunction(2,2,[0,1,1,0]);
< Boolean function of 2 variables >
gap> te:=BooleanFunctionBySTE(f);
[  ]
```

3.2.7 PDBooleanFunctionBySTE

▷ `PDBooleanFunctionBySTE(`*Func*`)` (function)

Let $f(x_1,\ldots,x_n)$ be a partially defined Boolean function. We denote by `x` the positions in truth vector, where f is undefined. Then f^{-1}`(x)` is the set of Boolean vectors of n variables on which the function is undefined. The sets $f^{-1}(0)$ and $f^{-1}(1)$ are defined in `KernelOfBooleanFunction` (2.1.9). The function f is called a *threshold function* if there is an n-dimensional real vector $w=(w_1,\ldots,w_n)$ and a real threshold T such that

$$a \in f^{-1}(1) \quad \Longrightarrow \quad a\cdot w^T \geq T,$$

$$a \in f^{-1}(0) \quad \Longrightarrow \quad a\cdot w^T < T,$$

where $a\cdot w^T$ is the classical inner (scalar) product.

For the partially defined Boolean function `Func` (presented as a string, where `x` presents the undefined values) the function `PDBooleanFunctionBySTE` returns a threshold element if `Func` can be realized by STE and empty list otherwise. The realization of the function `PDBooleanFunctionBySTE` is based on the algorithm, proposed in [GPR83].

Example

```
gap> f:="1x001x0x";
"1x001x0x"
gap> te:=PDBooleanFunctionBySTE(f);
< threshold element with weight vector [ -1, -2, -3 ] and threshold -1 >
gap> Display(te);
Weight vector = [ -1, -2, -3 ], Threshold = -1.
Threshold Element realizes the function f :
```

```
Boolean function of 3 variables.
[ 0, 0, 0 ] || 1
[ 0, 0, 1 ] || 0
[ 0, 1, 0 ] || 0
[ 0, 1, 1 ] || 0
[ 1, 0, 0 ] || 1
[ 1, 0, 1 ] || 0
[ 1, 1, 0 ] || 0
[ 1, 1, 1 ] || 0
Sum of Products:[ 0, 4 ]
```

3.3 Iterative Training Methods

Thelma also provides a few iterative methods for threshold element training.

3.3.1 ThresholdElementTraining

▷ ThresholdElementTraining(*ThrEl, Step, Func, Max_Iter*) (function)

This is a basic iterative method for the perceptron training [Ros58]. For the threshold element ThrEl (which is an arbitrary threshold element for the first iteration), the positive integer Step (the value on which we change parameters while training the threshold element), the Boolean function Func and the positive integer Max_Iter - the maximal number of iterations, the function ThresholdElementTraining returns a threshold element, realizing Func (if such threshold element exists).

Example

```
gap> f:=LogicFunction(2,2,[0,0,0,1]);
< Boolean function of 2 variables >
gap> te1:=RandomThresholdElement(2,-2,2);
< threshold element with weight vector [ 0, -1 ] and threshold 0 >
gap> Display(OutputOfThresholdElement(te1));
Boolean function of 2 variables.
[ 0, 0 ] || 1
[ 0, 1 ] || 0
[ 1, 0 ] || 1
[ 1, 1 ] || 0
gap> te2:=ThresholdElementTraining(te1,1,f,100);
< threshold element with weight vector [ 2, 1 ] and threshold 3 >
gap> Display(OutputOfThresholdElement(te2));
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 0
[ 1, 0 ] || 0
[ 1, 1 ] || 1
```

3.3.2 ThresholdElementBatchTraining

▷ `ThresholdElementBatchTraining(`*`ThrEl, Step, Func, Max_Iter`*`)` (function)

For the threshold element `ThrEl` (which is an arbitrary threshold element for the first iteration), the positive integer `Step` (the value on which we change parameters while training the threshold element), the Boolean function `Func`, and the positive integer `Max_Iter` - the maximal number of iterations, the function `ThresholdElementTraining` returns a threshold element, realizing `Func` (if such threshold element exists) via batch training.

Example

```
gap> f:=LogicFunction(2,2,[0,0,0,1]);
< Boolean function of 2 variables >
gap> te1:=RandomThresholdElement(2,-2,2);
< threshold element with weight vector [ 0, 2 ] and threshold 2 >
gap> Display(OutputOfThresholdElement(te1));
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 1
[ 1, 0 ] || 0
[ 1, 1 ] || 1
gap> te2:=ThresholdElementBatchTraining(te1,1,f,100);
< threshold element with weight vector [ 2, 2 ] and threshold 3 >
gap> Display(OutputOfThresholdElement(te2));
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 0
[ 1, 0 ] || 0
[ 1, 1 ] || 1
```

3.3.3 WinnowAlgorithm

▷ `WinnowAlgorithm(`*`Func, Step, Max_Iter`*`)` (function)

A Boolean function $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ which can be presented in the following form:

$$f(x_1,\ldots,x_n) = x_{i_1} \vee \cdots \vee x_{i_k}, \qquad (k \leq n)$$

is called a *monotone disjunction*, i.e. it is a disjunction in which no variable appears negated.

If the given Boolean function f is a monotone disjunction, the *Winnow algorithm* is more efficient than the classical Perceptron training algorithm [Lit88].

For the Boolean function `Func`, which is a monotone disjunction, `WinnowAlgorithm` returns either a threshold element realizing `Func` or [] if `Func` is not trainable by `WinnowAlgorithm`. The positive ingetger `Step` which is not equal to 1 defines the value on which we change parameters while running the algorithm and the positive integer `Max_Iter` defines the maximal number of iterations.

Example

```
gap> x:=Indeterminate(GF(2),"x");;
gap> y:=Indeterminate(GF(2),"y");;
gap> pol:=x*y+x+y;;
```

```
gap> f:=PolynomialToBooleanFunction(pol,2);
< Boolean function of 2 variables >
gap> Display(f);
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 1
[ 1, 0 ] || 1
[ 1, 1 ] || 1
gap> te:=WinnowAlgorithm(f,2,100);
< threshold element with weight vector [ 1, 1 ] and threshold 1 >
gap> Display(OutputOfThresholdElement(te));
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 1
[ 1, 0 ] || 1
[ 1, 1 ] || 1
```

3.3.4 Winnow2Algorithm

▷ `Winnow2Algorithm(`*Func, Step, Max_Iter*`)` (function)

For any $X \subseteq \mathbb{Z}_2^n$ and for any δ satisfying $0 < \delta \leq 1$ let $F(X,\delta)$ be the class of functions from X to $\mathbb{Z}_2^n$. Assume that $F(X,\delta)$ satisfies the following condition:

for each $f \in F(X,\delta)$ there exist $\mu_1,\ldots,\mu_n \geq 0$ such that for all $(x_1,\ldots,x_n) \in X$

$$\sum_{i=1}^{n} \mu_i x_i \geq 1, \quad \text{if} \quad f(x_1,\ldots,x_n) = 1$$

and

$$\sum_{i=1}^{n} \mu_i x_i \leq 1, \quad \text{if} \quad f(x_1,\ldots,x_n) = 0.$$

In other words, the inverse images of 0 and 1 are linearly separable with a minimum separation that depends on δ. `Winnow2` algorithm is designed for training this class of the Boolean functions [Lit88].

For the Boolean function `Func` from the class of Boolean functions which is described above, the function `Winnow2Algorithm` returns either a threshold element which realizes `Func` or [] if `Func` is not trainable by `Winnow2Algorithm`. The positive integer `Step` which is not equal to 1 defines the value on which we change parameters while running the algorithm.

Example

```
gap> ##Conjunction can not be trained by Winnow algorithm.
gap> x:=Indeterminate(GF(2),"x");;
gap> y:=Indeterminate(GF(2),"y");;
gap> pol:=x*y;;
gap> f:=PolynomialToBooleanFunction(pol,2);
< Boolean function of 2 variables >
gap> te:=WinnowAlgorithm(f,2,100);
[  ]
gap> ## But in the case of Winnow2 we can obtain the desirable result.
gap> te:=Winnow2Algorithm(f,2,100);
< threshold element with weight vector [ 1/2, 1/2 ] and threshold 1 >
gap> Display(te);
```

```
Weight vector = [ 1/2, 1/2 ], Threshold = 1.
Threshold Element realizes the function f :
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 0
[ 1, 0 ] || 0
[ 1, 1 ] || 1
Sum of Products:[ 3 ]
```

3.3.5 STESynthesis

▷ STESynthesis(*Func*) (function)

The function STESynthesis is based on the algorithm proposed in [Der65]. In each iteration we perturb an $n+1$-dimensional weight-threshold vector in such manner that the distance between the given vector and a desired weight-threshold vector, if such vector exists, is reduced. So if the Boolean function Func is STE-realizable, then this procedure will eventually yield an acceptable weight-threshold vector. Otherwise iteration process will eventually enter a limit cycle and the execution of STE_Synthesis will be stopped.

For the Boolean function Func the function STESynthesis returns a threshold element if Func is STE-realizable or an empty list otherwise.

Example

```
gap> f:=x*y+x+y;;
gap> x:=Indeterminate(GF(2),"x");;
gap> y:=Indeterminate(GF(2),"y");;
gap> pol:=x*y+x+y;;
gap> f:=PolynomialToBooleanFunction(pol,2);;
gap> te:=STESynthesis(f);
< threshold element with weight vector [ 2, 2 ] and threshold 1 >
gap> Display(te);
Weight vector = [ 2, 2 ], Threshold = 1.
Threshold Element realizes the function f :
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 1
[ 1, 0 ] || 1
[ 1, 1 ] || 1
Product of Sums:[ 0 ]
```

Chapter 4

Networks of Threshold Elements

Not all Boolean functions can be realized by a single threshold element. However, all of them can be realized by a multi-layered network of threshold elements, with a number of threshold elements on a first layer and conjunction or a disjunction on the second layer. In this chapter we will decribe some functions regarding such networks.

4.1 Basic Operations

In this section we describe some operations, similar to the ones described in Section 3.1.

4.1.1 NeuralNetwork

▷ `NeuralNetwork(`*`InnerLayer, OuterLayer`*`)` (function)

For the list of threshold elements `InnerLayer` and the Boolean variable `OuterLayer`, which can be either `true` (for disjunction), `false` (for conjunction), or `fail` (if there is only one layer) the function `NeuralNetwork` returns a neural network built from this inputs.

Example

```
gap> te1:=ThresholdElement([1,1],1);
< threshold element with weight vector [ 1, 1 ] and threshold 1 >
gap> te2:=ThresholdElement([-1,-2],-2);
< threshold element with weight vector [ -1, -2 ] and threshold -2 >
gap> inner:=[te1,te2];
[ < threshold element with weight vector [ 1, 1 ] and threshold 1 >,
  < threshold element with weight vector [ -1, -2 ] and threshold -2 > ]
gap> nn:=NeuralNetwork(inner,false);
< neural network with
2 threshold elements on inner layer and conjunction on outer level >
gap> Display(last);
Inner Layer:
[ [[ 1, 1 ], 1], [[ -1, -2 ], -2] ]
Outer Layer: conjunction
Neural Network realizes the function f :
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 1
```

```
[ 1, 0 ] || 1
[ 1, 1 ] || 0
Sum of Products:[ 1, 2 ]
```

4.1.2 IsNeuralNetwork

▷ IsNeuralNetwork(*Obj*) (function)

For the object Obj the function IsNeuralNetwork returns true if Obj is a neural network (see NeuralNetwork (4.1.1)), and false otherwise.

Example

```
gap> ## Consider the neural network <C>nn</C> from the previous example.
gap> IsNeuralNetwork(nn);
true
```

4.1.3 OutputOfNeuralNetwork

▷ OutputOfNeuralNetwork(*NNetwork*) (function)

For the neural network NNetwork the function OutputOfNeuralNetwork returns the Boolean function, realized by NNetwork.

Example

```
gap> f:=OutputOfNeuralNetwork(nn);
< Boolean function of 2 variables >
gap> Display(last);
Boolean function of 2 variables.
[ 0, 0 ] || 0
[ 0, 1 ] || 1
[ 1, 0 ] || 1
[ 1, 1 ] || 0
```

4.2 Networks of Threshold Elements

In this section we consider the networks of threshold elements.

4.2.1 BooleanFunctionByNeuralNetwork

▷ BooleanFunctionByNeuralNetwork(*Func*) (function)

For the Boolean function Func the function BooleanFunctionByNeuralNetwork returns a two-layered neural network, which realizes Func (see NeuralNetwork (4.1.1)). The realization of this function is based on the algorithm proposed in [GPR83].

Example

```
gap> x:=Indeterminate(GF(2),"x");;
gap> y:=Indeterminate(GF(2),"y");;
gap> z:=Indeterminate(GF(2),"z");;
gap> f:=PolynomialToBooleanFunction(x*y+z,3);
< Boolean function of 3 variables >
gap> nn:=BooleanFunctionByNeuralNetwork(f);
< neural network with
2 threshold elements on inner layer and disjunction on outer level >
gap> Display(last);
Inner Layer:
[ [[ -1, -2, 4 ], 2], [[ 1, 2, -3 ], 3] ]
Outer Layer: disjunction
Neural Network realizes the function f :
Boolean function of 3 variables.
[ 0, 0, 0 ] || 0
[ 0, 0, 1 ] || 1
[ 0, 1, 0 ] || 0
[ 0, 1, 1 ] || 1
[ 1, 0, 0 ] || 0
[ 1, 0, 1 ] || 1
[ 1, 1, 0 ] || 1
[ 1, 1, 1 ] || 0
Sum of Products:[ 1, 3, 5, 6 ]
```

4.2.2 BooleanFunctionByNeuralNetworkDASG

▷ `BooleanFunctionByNeuralNetworkDASG(`*Func*`)` (function)

For the Boolean function `Func` the function `BooleanFunctionByNeuralNetworkDASG` returns a two-layered neural network which realizes `Func` (see `NeuralNetwork` (4.1.1)). The realization of this function is based on decomposition of `Func` by the non-unate variables with the biggest influence. The DASG algorithm (DASG - Decomposition Algorithm for Synthesis and Generalization) was proposed in [SJF08], however we use a slightly modified version of this algorithm.

Example

```
gap> f:=LogicFunction(3,2,[0,0,0,0,0,1,1,0]);
< Boolean function of 3 variables >
gap> nn:=BooleanFunctionByNeuralNetworkDASG(f);
< neural network with 2 threshold elements on inner layer and conjunction on outer level >
gap> Display(last);
Inner Layer:
[ [[ 1, 4, 2 ], 3], [[ 1, -4, -2 ], -3] ]
Outer Layer: conjunction
Neural Network realizes the function f :
Boolean function of 3 variables.
[ 0, 0, 0 ] || 0
[ 0, 0, 1 ] || 0
[ 0, 1, 0 ] || 0
```

```
[ 0, 1, 1 ] || 0
[ 1, 0, 0 ] || 0
[ 1, 0, 1 ] || 1
[ 1, 1, 0 ] || 1
[ 1, 1, 1 ] || 0
Sum of Products:[ 5, 6 ]
```

Chapter 5

Multi-Valued Threshold Elements

5.1 Threshold Elements over $GF(p^k)$

Let $F = GF(p^k)$ be a Galois field with primitive element ε and let F^* be the multiplicative group of F. Let us consider all proper subgroups of F^*:

$$C_{j_1} = \langle \sigma_1 \mid \sigma_1^{j_1} = 1 \rangle, \ldots, C_{j_t} = \langle \sigma_t \mid \sigma_n^{j_t} = 1 \rangle,$$

where $\sigma_i = \varepsilon^{\frac{|F^*|}{j_i}}$ $(i = 1, \ldots, t)$are the generators of the corresponding cyclic groups. Denote $T = \{j_1, \ldots, j_t\}$. Let $G_n = C_{k_1} \otimes \ldots \otimes C_{k_n}$ be a direct product of cyclic groups C_{k_i} ($k_i \in T$,$i = 1, \ldots, n$).

A *discrete function* of n variables over a finite field F is a mapping $f : G_n \to C_q$, $q \in T$, $C_q = \langle \sigma \mid \sigma^q = 1 \rangle$, $\sigma = \varepsilon^{\frac{|F^*|}{q}}$.

We define the function $FSign\xi$ in the following way [Gec10]:

$$\forall \xi \in F^* : FSign\xi = \sigma^j, \text{ if } \frac{j|F^*|}{q} \leq \deg\xi < \frac{(j+1)|F^*|}{q},$$

in which $\deg\xi$ is obtained from the equality $\xi = \varepsilon^{\deg\xi}$, $j \in \{1, \ldots, q-1\}$.

Let $(w_1, \ldots, w_n; T) \in F^{n+1}$. For all $\overline{g} = (x_1, \ldots, x_n) \in G_n$ (i.e. $x_j \in C_{k_j}$) we define

$$w(\overline{g}) = \sum_{i=1}^{n} w_i x_i + T \in F.$$

A *neural element* with structure vector $(w_1, \ldots, w_n; T) \in F^{n+1}$ is a logic device that realizes the function $FSign(w(\overline{g}))$ for all $\overline{g} \in G_n$.

Discrete function f which is relizable by a single neural element over the field F is called *neurofunction*.

5.1.1 MVThresholdElement

▷ `MVThresholdElement(`*`Structure, Dimensions, Field`*`)` (function)

For the two-element list `Structure`, in which the first element is a vector over the field `Field`, and the second element is an element of the `Field`, a list of positive integers `Dimensions` (or an integerm if all the dimensions are equal) and a Galois field `Field` the function `MVThresholdElement` returns a multi-valued threshold element with the number of inputs equal to the length of the first element of `Structure` list.

Example

```
gap> F:=GF(13);;
gap> st:=[[Z(13)^5,Z(13)^7],Z(13)^4];;
gap> dim:=[2,3,3];;
gap> mvte:=MVThresholdElement(st,dim,F);
< multivalued threshold element over GF(13) with structure [[ 6, 11 ], 3] and
  dimension vector [ 2, 3, 3 ] >
gap> Display(mvte);
Structure vector = [[ 6, 11 ], 3]
Dimension vector = [[ 2, 3, 3 ]]
Field: GF(13)
Multi-Valued Threshold Element realizes the function f :
[ 1, 1 ]  ||  9
[ 1, 3 ]  ||  3
[ 1, 9 ]  ||  1
[ 12, 1 ]  ||  1
[ 12, 3 ]  ||  1
[ 12, 9 ]  ||  9
gap> ## If all dimensions in dimension vector are equal, the user can enter
just this number.
gap> F:=GF(5);;
gap> st:=[[Z(5)^0,Z(5)^0],Z(5)^2];;
gap> dim:=2;;
gap> mvte:=MVThresholdElement(st,dim,F);
< multivalued threshold element over GF(5) with structure [[ 1, 1 ], 4] and
  dimension vector [ 2, 2, 2 ] >
gap> Display(mvte);
Structure vector = [[ 1, 1 ], 4]
Dimension vector = [[ 2, 2, 2 ]]
Field: GF(5)
Multi-Valued Threshold Element realizes the function f :
[ 1, 1 ]  ||  1
[ 1, 4 ]  ||  4
[ 4, 1 ]  ||  4
[ 4, 4 ]  ||  1
```

The function `Display` outputs the stucture of the given threshold element `mvte` and the truth table in given alphabet. In the previous example `mvte` realizes a three-valued function in $\{1,3,9\}$ alphabet over GF(13).

5.1.2 IsMVThresholdElement

▷ `IsMVThresholdElement(`*Obj*`)` (function)

For the object `Obj` the function `IsMVThresholdElement` returns `true` if `Obj` is a multi-valued threshold element (see `MVThresholdElement` (5.1.1)), and `false` otherwise.

Example

```
gap> IsMVThresholdElement(mvte);
```

```
true
```

5.1.3 OutputOfMVThresholdElement

▷ `OutputOfMVThresholdElement(`*MVThrEl*`)` (function)

For the multi-valued threshold element `MVThrEl` the function `OutputOfMVThresholdElement` returns the truth vector of the multi-valued Boolean function, realized by `MVThrEl`.

Example

```
gap> F:=GF(13);;
gap> st:=[[Z(13)^5,Z(13)^7],Z(13)^4];;
gap> dim:=[2,3,3];;
gap> mvte:=MVThresholdElement(st,dim,F);
< multivalued threshold element over GF(13) with structure [[ 6, 11 ], 3] and
  dimension vector [ 2, 3, 3 ] >
gap> f:=OutputOfMVThresholdElement(mvte);
< logic function of 2 variables and dimension vector [ 2, 3, 3 ]>
gap> Display(f);
[ 0, 0 ] || 2
[ 0, 1 ] || 1
[ 0, 2 ] || 0
[ 1, 0 ] || 0
[ 1, 1 ] || 0
[ 1, 2 ] || 2
```

5.1.4 StructureOfMVThresholdElement

▷ `StructureOfMVThresholdElement(`*MVThrEl*`)` (function)

For the multi-valued threshold element `MVThrEl` the function `StructureOfMVThresholdElement` returns the structure vector of `MVThrEl` (see `MVThresholdElement` (5.1.1)).

Example

```
gap> StructureOfMVThresholdElement(mvte);
[ [ Z(13)^5, Z(13)^7 ], Z(13)^4 ]
```

5.1.5 MVBooleanFunctionBySTE

▷ `MVBooleanFunctionBySTE(`*Func, Dim, F*`)` (function)

For the given multi-valued function `Func` and the prime field `F` the function `MVBooleanFunctionBySTE` returns the multi-valued threshold element over `F` if `Func` is realizable and an empty list otherwise. The algorithm realizing this function was proposed in [Gec10].

Example

```
gap> f:=LogicFunction(2,2,[0,1,1,0]);
< Boolean function of 2 variables >
gap> mvte:=MVBooleanFunctionBySTE(f,GF(3));
[  ]
gap> mvte:=MVBooleanFunctionBySTE(f,GF(5));
< multivalued threshold element over GF(5) with structure [[ 1, 1 ], 4] and
  dimension vector [ 2, 2, 2 ] >
gap> Display(last);
Structure vector = [[ 1, 1 ], 4]
Dimension vector = [[ 2, 2, 2 ]]
Field: GF(5)
Multi-Valued Threshold Element realizes the function f :
[ 1, 1 ]  ||  1
[ 1, 4 ]  ||  4
[ 4, 1 ]  ||  4
[ 4, 4 ]  ||  1
gap> ## Consider an example if dimensions are presented as a list.
gap> f:=LogicFunction(2,[2,3,3],[0,0,1,1,2,2]);
< logic function of 2 variables and dimension vector [ 2, 3, 3 ]>
gap> mvte:=MVBooleanFunctionBySTE(f,GF(13));
< multivalued threshold element over GF(13) with structure [[ 12, 10 ], 5]
  and dimension vector [ 2, 3, 3 ] >
gap> Display(last);
Structure vector = [[ 12, 10 ], 5]
Dimension vector = [[ 2, 3, 3 ]]
Field: GF(13)
Multi-Valued Threshold Element realizes the function f :
[ 1, 1 ]  ||  1
[ 1, 3 ]  ||  1
[ 1, 9 ]  ||  3
[ 12, 1 ]  ||  3
[ 12, 3 ]  ||  9
[ 12, 9 ]  ||  9
```

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